Diagonalización

6.2

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(*) $$p_H(x) = \begin{vmatrix} 1-x & 1 & 1 & 1 \\ 1 & 1-x & -1 & -1 \\ 1 & -1 & 1-x & -1 \\ 1 & -1 & -1 & 1-x \end{vmatrix} \underset{\substack{F_3 - F_2 \\ F_4 - F_2}}{\overset{F_1 - (1-x)F_2}{=\!=}} \begin{vmatrix} 0 & 1-(1-x)^2 & 2-x & 2-x \\ 1 & 1-x & -1 & -1 \\ 0 & -2+x & 2-x & 0 \\ 0 & -2+x & 0 & 2-x \end{vmatrix}$$

$$= - \begin{vmatrix} x(2-x) & 2-x & 2-x \\ -2+x & 2-x & 0 \\ -2+x & 0 & 2-x \end{vmatrix} = -(2-x)^3 \begin{vmatrix} x & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \overset{F_3-F_2}{=}$$

$$= -(2-x)^3 (x+1+1) = -(2+x)(2-x)^3$$

(*) $\sigma_H = \{2, -2\}$ $\quad m(2) = 3 \quad m(-2) = 1$

(*) $\dim V_{-2} = m(-2) = 1$ (se cumple siempre)

(*) $V_2 = \{ \vec{x} \in \mathbb{R}^4 \text{ tq } (H - 2I_4)\vec{x} = \vec{0} \}$

$$H - 2I_4 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix} \underset{\substack{F_3 + F_1 \\ F_4 + F_1}}{\overset{F_2 + F_1}{\sim}} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Así que $V_2 = \{(x,y,z,t) \in \mathbb{R}^4$ tq $-x+y+z+t=0\}$ y

$\dim V_2 = 4 - rg\, H - 2I_4 = 4-1=3 = m(2)$

* Puesto que $\dim V_2 = m(2)$ y $\dim V_{-2} = m(-2)$ se tiene que H es diagonalizable.

* Buscamos una base de V_2, es decir, 3 vectores l. independientes que verifiquen la ecuación $-x+y+z+t=0$:

$$\beta_{V_2} = \{(1,1,0,0), (1,0,1,0), (1,0,0,1)\}$$

* Buscamos ahora una base de V_{-2}, un vector no nulo que pertenezca a $V_{-2} = \{\vec{x} \in \mathbb{R}^4$ tq $(H+2I_4)\vec{x} = \vec{0}\}$

$$H+2I_4 = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix} \underset{\substack{F_4-F_2 \\ F_1-3F_2}}{\overset{F_3-F_2}{\sim}} \begin{pmatrix} 0 & -8 & 4 & 4 \\ 1 & 3 & -1 & -1 \\ 0 & -4 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \underset{\substack{\frac{1}{4}F_3 \\ \frac{1}{4}F_4}}{\overset{\frac{1}{4}F_1}{\sim}}$$

$$\sim \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \underset{F_4-F_3}{\overset{F_1-2F_3}{\sim}} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \overset{F_4-F_1}{\sim}$$

$$\sim \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Así que $V_{-2} = \{(x,y,z,t)$ tq $x+3y-z-t = y-t = z-t=0\}$

$$\beta_{V_{-2}} = \{(-1,1,1,1)\}$$

(*) Finalmente $D = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ y $P = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

y $\quad P^{-1}HP = D$

(2) Como $P^{-1}HP = D$ entonces

$$(P^{-1}HP)^2 = P^{-1}H\cancel{P}\cancel{P^{-1}}HP = P^{-1}H^2P = D^2 = \begin{pmatrix} 4 & & & 0 \\ & 4 & & \\ & & 4 & \\ 0 & & & 4 \end{pmatrix}$$

y entonces H^2 es diagonalizable

(*) Usando $P^{-1}HP = D$ tenemos:

$$(P^{-1}HP)^{-1} = P^{-1}H^{-1}(P^{-1})^{-1} = P^{-1}H^{-1}P = D^{-1} =$$

$$= \begin{pmatrix} -1/2 & & & 0 \\ & 1/2 & & \\ & & 1/2 & \\ 0 & & & 1/2 \end{pmatrix} \Rightarrow H^{-1} \text{ es diagonalizable.}$$

Extra Diagonaliza la matriz

$$A = \begin{pmatrix} -1 & 5 & 3 & -9 \\ -3 & 7 & 3 & -9 \\ -3 & 5 & 5 & -9 \\ -2 & 2 & 6 & -8 \end{pmatrix}$$

Resolución:

$$P_A(x) = \begin{vmatrix} -1-x & 5 & 3 & -9 \\ -3 & 7-x & 3 & -9 \\ -3 & 5 & 5-x & -9 \\ -2 & 2 & 6 & -8-x \end{vmatrix} \underset{F_2-F_3}{\overset{F_1-F_2}{=}} \begin{vmatrix} 2-x & -2+x & 0 & 0 \\ 0 & 2-x & -2+x & 0 \\ -3 & 5 & 5-x & -9 \\ -2 & 2 & 6 & -8-x \end{vmatrix}$$

$$= (2-x)^2 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -3 & 5 & 5-x & -9 \\ -2 & 2 & 6 & -8-x \end{vmatrix} \overset{C_2+C_3}{=} \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -3 & 10-x & 5-x & -9 \\ -2 & 8 & 6 & -8-x \end{vmatrix} (2-x)^2 =$$

$$= (2-x)^2 \begin{vmatrix} 1 & -1 & 0 \\ -3 & 10-x & -9 \\ -2 & 8 & -8-x \end{vmatrix} \overset{C_1+C_2}{=} \begin{vmatrix} 0 & -1 & 0 \\ 7-x & 10-x & -9 \\ 6 & 8 & -8-x \end{vmatrix} (2-x)^2$$

$$= (2-x)^2 \begin{vmatrix} 7-x & -9 \\ 6 & -8-x \end{vmatrix} = (2-x)^2 \left[-56+x+x^2+54\right] =$$

$$= (2-x)^2 (x^2+x-2) = 0 \left\langle \begin{array}{l} (2-x)^2 = 0 \Rightarrow x = 2 \\ x^2+x-2 = 0 \Rightarrow x = \dfrac{-1 \pm \sqrt{1+8}}{2} = \left\langle \begin{array}{l} 1 \\ -2 \end{array} \right. \end{array} \right.$$

Así que $\sigma_A = \{2, 1, -2\}$ y $m(2) = 2$, $m(1) = m(-2) = 1$

* Como $m(-2) = m(1) = 1$ se tiene que

$$\dim V_{-2} = m(-2) \quad \text{y}$$

$$\dim V_1 = m(1)$$

* Estudiamos ahora si $m(2) = \dim V_2$.

$$V_2 = \{\vec{x} \in \mathbb{R}^4 : (A - 2I_4)\,\vec{x} = \vec{0}\}$$

$$A - 2I_4 = \begin{pmatrix} -3 & 5 & 3 & -9 \\ -3 & 5 & 3 & -9 \\ -3 & 5 & 3 & -9 \\ -2 & 2 & 6 & -10 \end{pmatrix} \underset{F_2 - F_3}{\overset{F_1 - F_3}{\sim}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 5 & 3 & -9 \\ -2 & 2 & 6 & -10 \end{pmatrix} \overset{\frac{1}{2}F_4}{\sim}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 5 & 3 & -9 \\ -1 & 1 & 3 & -5 \end{pmatrix} \overset{F_3 - F_4}{\sim} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & -4 \\ -1 & 1 & 3 & -5 \end{pmatrix} \overset{-\frac{1}{2}F_3}{\sim} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & -2 & 0 & 2 \\ -1 & 1 & 3 & -5 \end{pmatrix}$$

$\dim V_2 = 4 - rg(A - 2I_4) = 4 - 2 = 2 = m(2)$, luego A es diagonalizable.

$$V_2 = \{(x, y, z, t) \in \mathbb{R}^4 : x - 2y + 2t = -x + y + 3z - 5t = 0\}$$

$$\beta_{V_2} = \{(0,3,4,3),\ (6,3,1,0)\}$$

(*) Buscamos ahora una base de V_{-2}.

$$V_{-2} = \{\vec{x} \in \mathbb{R}^4 \text{ tq } (A+2I_4)\vec{x} = \vec{0}\}$$

$$A+2I_4 = \begin{pmatrix} 1 & 5 & 3 & -9 \\ -3 & 9 & 3 & -9 \\ -3 & 5 & 7 & -9 \\ -2 & 2 & 6 & -6 \end{pmatrix} \underset{\substack{F_2-F_3 \\ \frac{1}{2}F_3}}{\overset{F_1-F_3}{\sim}} \begin{pmatrix} 4 & 0 & -4 & 0 \\ 0 & 4 & -4 & 0 \\ -3 & 5 & 7 & -9 \\ -1 & 1 & 3 & -3 \end{pmatrix} \underset{\frac{1}{4}F_2}{\overset{\frac{1}{4}F_1}{\sim}}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -3 & 5 & 7 & -9 \\ -1 & 1 & 3 & -3 \end{pmatrix}$$

ponemos las 3 primeras porque son indep. y $rg(A+2I_4)=3$

$$V_{-2} = \{(x,y,z,t) : x-z = y-z = -3x+5y+7z-9t = 0\}$$

$$\beta_{V_{-2}} = \{(1,1,1,1)\}$$

(*) Buscamos una base de $V_1 = \{\vec{x} \in \mathbb{R}^4 \text{ tq } (A-I)\vec{x} = \vec{0}\}$

$$A+I_4 = \begin{pmatrix} -2 & 5 & 3 & -9 \\ -3 & 6 & 3 & -9 \\ -3 & 5 & 4 & -9 \\ -2 & 2 & 6 & -9 \end{pmatrix} \underset{F_4-F_1}{\overset{F_3-F_2}{\sim}} \begin{pmatrix} -2 & 5 & 3 & -9 \\ -3 & 6 & +3 & -9 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{pmatrix} \underset{\frac{1}{3}F_2}{\overset{F_4-3F_3}{\sim}}$$

$$\sim \begin{pmatrix} -2 & 5 & 3 & -9 \\ -1 & 2 & 1 & -3 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_1 = \{(x,y,z,t) : -2x+5y+3z-9t = -y+z = -x+2y+z-3t = 0\}$$

$$\beta_{V_1} = \{(3,3,3,2)\}$$

Así que:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad y \quad P = \begin{pmatrix} 3 & 1 & 0 & 6 \\ 3 & 1 & 3 & 3 \\ 3 & 1 & 4 & 1 \\ 2 & 1 & 3 & 0 \end{pmatrix}$$