Cálculo diferencial de funcions de varias variables

9.1

$$f(x,y) = \begin{cases} \dfrac{x \operatorname{sen} y - y \operatorname{sen} x}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$$

Calculemos $\dfrac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(\operatorname{sen} y - y \cos x)(x^2+y^2) - 2x(x \operatorname{sen} y - y \operatorname{sen} x)}{(x^2+y^2)^2} =$$

$$= \frac{(y^2-x^2)\operatorname{sen} y - y(x^2+y^2)\cos x + 2xy \operatorname{sen} x}{(x^2+y^2)^2} \quad \text{si } (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f((0,0)+h(1,0)) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} =$$

$$= \lim_{h\to 0} \frac{h \operatorname{sen} 0 - 0 \operatorname{sen} h}{h^3} = 0$$

Calculemos $\dfrac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y}(x,y) = \frac{(x\cos y - \operatorname{sen} x)(x^2+y^2) - 2y(x\operatorname{sen} y - y\operatorname{sen} x)}{(x^2+y^2)^2} =$$

$$= \frac{(y^2-x^2)\operatorname{sen} x + x(x^2+y^2)\cos y - 2xy\operatorname{sen} y}{(x^2+y^2)^2} \quad \text{si } (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f((0,0)+h(0,1)) - f(0,0)}{h} = \lim_{h\to 0}\frac{f(0,h)}{h} =$$

$$= \lim_{h\to 0}\frac{0\operatorname{sen} h - h\operatorname{sen} 0}{h^3} = 0$$

Calculemos $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}((0,0)+h(0,1)) - \frac{\partial f}{\partial x}(0,0)}{h} =$$

$$= \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(0,h)}{h} = \lim_{h\to 0}\frac{h^2\operatorname{sen} h - h^3}{h^5} =$$

$$= \lim_{h\to 0}\frac{h^2\left(h - \frac{h^3}{3!}\right) - h^3 + o(h^5)}{h^5} = \lim_{h\to 0}\frac{-\frac{h^5}{3!} + o(h^5)}{h^5} =$$

$$= \frac{-1}{6}$$

Calculemos $\boxed{\dfrac{\partial^2 f}{\partial y \partial x}(0,0)}$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}((0,0)+h(1,0)) - \frac{\partial f}{\partial y}(0,0)}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0)}{h} = \lim_{h\to 0} \frac{-h^2 \operatorname{sen} h + h^3}{h^5} =$$

$$= \lim_{h\to 0} \frac{-h^2\left(h - \frac{h^3}{3!}\right) + h^3 + o(h^5)}{h^5} = \lim_{h\to 0} \frac{+\frac{h^5}{3!} + o(h^5)}{h^5} =$$

$$= \frac{1}{6}$$

9.2

$f(x,y) = (x^2y^4,\ x^3y^3 + 4xy^2)$

$g(x,y) = (x \operatorname{sen} y,\ y \operatorname{sen} x)$

$F = g \circ f$

$$Jg(x,y) = \begin{pmatrix} \operatorname{sen} y & x\cos y \\ y\cos x & \operatorname{sen} x \end{pmatrix}$$

$$Jf(x,y) = \begin{pmatrix} 2xy^4 & 4y^3x^2 \\ 3x^2y^3 + 4y^2 & 3y^2x^3 + 8xy \end{pmatrix}$$

$$JF(2,-1) = Jg(f(2,-1)) \cdot Jf(2,-1) = Jg(4,0)\,Jf(2,-1)$$

$$= \begin{pmatrix} 0 & 4 \\ 0 & \operatorname{sen} 4 \end{pmatrix} \begin{pmatrix} 4 & -16 \\ -8 & 8 \end{pmatrix} = \begin{pmatrix} -32 & 32 \\ -8\operatorname{sen} 4 & 8\operatorname{sen} 4 \end{pmatrix}$$

9.3

$f(t) = (t^2,\ 3t-1,\ 1-t^2)$

$g(x,y,z) = (x^2 - y - zx,\ y^2 + xy + z^2)$

$F = g \circ f$

$JF(-1) = J\,g\circ f(-1) = J\,g(f(-1)) \cdot Jf(-1) = Jg(1,-4,0)\,Jf(-1)$

$Jf(t) = \begin{pmatrix} 2t \\ 3 \\ -2t \end{pmatrix} \qquad Jg(x,y,z) = \begin{pmatrix} 2x-z & -1 & -x \\ y & 2y+x & 2z \end{pmatrix}$

$JF(-1) = \begin{pmatrix} 2 & -1 & -1 \\ -4 & -7 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ -13 \end{pmatrix}$

9.4

$$F(x,y) = f(g(x)k(y),\ g(x)+h(y))$$

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \underline{f: \mathbb{R}^2 \longrightarrow \mathbb{R}}$$

$$G: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longrightarrow (g(x)k(y),\ g(x)+h(y))$$

$$F(x,y) = f \circ G(x,y)$$

$$JF(x,y) = J f\circ G(x,y) = Jf(G(x,y)) \cdot JG(x,y)$$

$$Jf(x,y) = \left(\frac{\partial f}{\partial x}(x,y),\ \frac{\partial f}{\partial y}(x,y)\right)$$

$$JG(x,y) = \begin{pmatrix} g'(x)k(y) & g(x)k'(y) \\ g'(x) & h'(y) \end{pmatrix}$$

$$JF(x,y) = \left(\frac{\partial f}{\partial x}(g(x)k(y), g(x)+h(y)),\ \frac{\partial f}{\partial y}(g(x)k(y), g(x)+h(y))\right) \cdot \begin{pmatrix} g'(x)k(y) & g(x)k'(y) \\ g'(x) & h'(y) \end{pmatrix} =$$

$$= \Bigg(\underbrace{\frac{\partial f}{\partial x}(G(x,y)) \cdot g'(x)k(y) + \frac{\partial f}{\partial y}(G(x,y)) \cdot g'(x)}_{\frac{\partial F}{\partial x}(x,y)},$$
$$\underbrace{\frac{\partial f}{\partial x}(G(x,y)) \cdot g(x)k'(y) + \frac{\partial f}{\partial y}(G(x,y)) \cdot h'(y)}_{\frac{\partial F}{\partial y}(x,y)} \Bigg)$$

9.5

$$F(x,y,z) = f(x^y, y^z, z^x)$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longrightarrow (x^y, y^z, z^x)$$

$$F(x,y,z) = f \circ G(x,y,z)$$

$$JF(x,y,z) = Jf(G(x,y,z)) \cdot JG(x,y,z) =$$

$$= \left(\frac{\partial f}{\partial x}(x^y, y^z, z^x), \frac{\partial f}{\partial y}(x^y, y^z, z^x), \frac{\partial f}{\partial z}(x^y, y^z, z^x) \right)$$

$$\begin{pmatrix} y\,x^{y-1} & x^y \log x & 0 \\ 0 & z\,y^{z-1} & y^z \log y \\ z^x \log z & 0 & x\,z^{x-1} \end{pmatrix} =$$

$$= \Bigg(\underbrace{\frac{\partial f}{\partial x}(G(x,y,z)) \cdot y\,x^{y-1} + \frac{\partial f}{\partial z}(G(x,y,z)) \cdot z^x \log z}_{\frac{\partial F}{\partial x}(x,y,z)},$$

$$\underbrace{\frac{\partial f}{\partial x}(G(x,y,z))\, x^y \log x + \frac{\partial f}{\partial y}(G(x,y,z))\, z\, y^{z-1}}_{\frac{\partial F}{\partial y}(x,y,z)},$$

$$\underbrace{\frac{\partial f}{\partial y}(G(x,y,z))\, y^z \log y + \frac{\partial f}{\partial z}(G(x,y,z)) \cdot x\, z^{x-1}}_{\frac{\partial F}{\partial z}(x,y,z)} \Bigg)$$

9.6 a) $F(x,y,z) = \int_0^{x+y+z} \operatorname{sen} t \, dt$

$g(x,y,z) = x+y+z$

$h(x) = \int_0^{x} \operatorname{sen} t \, dt$

$F(x,y,z) = h \circ g(x,y,z)$

$JF(x,y,z) = Jh(g(x,y,z)) \cdot Jg(x,y,z) =$

$= \operatorname{sen}(g(x,y,z)) \cdot (1, 1, 1) =$

$= \Big(\underbrace{\operatorname{sen}(x+y+z)}_{\frac{\partial F}{\partial x}(x,y,z)}, \underbrace{\operatorname{sen}(x+y+z)}_{\frac{\partial F}{\partial y}(x,y,z)}, \underbrace{\operatorname{sen}(x+y+z)}_{\frac{\partial F}{\partial z}(x,y,z)}\Big)$

9.6.b $F(x,y,z) = \int_0^{xyz} t \operatorname{sen} t \, dt$

$$\frac{\partial F}{\partial x}(x,y,z) = xyz \operatorname{sen}(xyz) \cdot yz = xy^2z^2 \operatorname{sen}(xyz)$$

$$\frac{\partial F}{\partial y}(x,y,z) = xyz \operatorname{sen}(xyz) \cdot xz = x^2 y z^2 \operatorname{sen}(x,y,z)$$

$$\frac{\partial F}{\partial z}(x,y,z) = x^2 y^2 z \operatorname{sen}(xyz)$$

9.6.c

$$F(x,y,z) = \int_{x^2+y^2}^{xyz} \operatorname{sen} t \, dt = -\int_0^{x^2+y^2} \operatorname{sen} t \, dt + \int_0^{xyz} \operatorname{sen} t \, dt$$

$$\frac{\partial F}{\partial x}(x,y,z) = yz \operatorname{sen}(xyz) - 2x \operatorname{sen}(x^2+y^2)$$

$$\frac{\partial F}{\partial y}(x,y,z) = xz \operatorname{sen}(xyz) - 2y \operatorname{sen}(x^2+y^2)$$

$$\frac{\partial F}{\partial z}(x,y,z) = xy \operatorname{sen}(xyz)$$

9.7 $\quad F(x,y) = f\Big(\overbrace{\frac{1}{y} - \frac{1}{x}}^{P}\Big)$

a) $¿\, x^2 \dfrac{\partial F}{\partial x}(x,y) + y^2 \dfrac{\partial F}{\partial y}(x,y) = 0\,?$ Sí

$$\parallel$$

$$x^2 f'(P)\cdot \frac{1}{x^2} + y^2 f'(P)\cdot \frac{-1}{y^2} = f'(P) - f'(P) = 0$$

b) $$¿\, \underbrace{2x^2y^2(x+y)\frac{\partial^2 F}{\partial x \partial y}(x,y) - x^4 \frac{\partial^2 F}{\partial x^2}(x,y) + y^4 \frac{\partial^2 F}{\partial y^2}(x,y)}_{\parallel \atop H} = 0\,?$$

$$\frac{\partial F}{\partial x}(x,y) = f'\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{1}{x^2} \quad ; \quad \frac{\partial F}{\partial y}(x,y) = f'\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{-1}{y^2}$$

$$\frac{\partial^2 F}{\partial x \partial y}(x,y) = f''\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{1}{x^2}\ \frac{-1}{y^2}$$

$$\frac{\partial^2 F}{\partial x^2}(x,y) = f''\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{1}{x^2}\cdot \frac{1}{x^2} + f'\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{-2}{x^3}$$

$$\frac{\partial^2 F}{\partial y^2}(x,y) = f''\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{-1}{y^2}\ \frac{-1}{y^2} + f'\left(\frac{1}{y} - \frac{1}{x}\right)\cdot \frac{2}{y^3}$$

$$H = 2x^2y^2(x+y)\frac{\partial^2 F}{\partial x \partial y}(x,y) - x^4 \frac{\partial^2 F}{\partial x^2}(x,y) + y^4 \frac{\partial^2 F}{\partial y^2}(x,y) =$$

$$= -2(x+y)f''(P) - f''(P) + f'(P)\,2x + f''(P) + 2y\,f'(P) =$$

$$= 2(x+y)\,f'\left(\frac{1}{y} - \frac{1}{x}\right) - 2(x+y)\,f''(P) = 2(x+y)\big(f'(P) - f''(P)\big) = 0$$

(9.8) $F(x,y) = \dfrac{f(y/x)}{x}$

a) $¿\, x \dfrac{\partial F}{\partial x}(x,y) + y \dfrac{\partial F}{\partial y}(x,y) + F(x,y) = 0\,?$ sí

$$x \cdot \frac{f'(y/x) \cdot \frac{-y}{x^2} \cdot x - f(y/x)}{x^2} + y \frac{1}{x} f'(y/x) \frac{1}{x} + \frac{f(y/x)}{x} =$$

$$= \cancel{-f'(y/x)\frac{y}{x^2}} \cancel{- \frac{1}{x} f(y/x)} + \cancel{\frac{y}{x^2} f'(y/x)} + \cancel{\frac{f(y/x)}{x}} = 0$$

b) $\dfrac{\partial F}{\partial x}(x,y) = -f'\left(\dfrac{y}{x}\right) \dfrac{y}{x^3} - \dfrac{1}{x^2} f\left(\dfrac{y}{x}\right)$

$\dfrac{\partial F}{\partial y}(x,y) = \dfrac{1}{x^2} f'(y/x)$

$\dfrac{\partial^2 F}{\partial x^2}(x,y) = f''\left(\dfrac{y}{x}\right) \cdot \dfrac{y^2}{x^5} + f'\left(\dfrac{y}{x}\right) \cdot \dfrac{3y}{x^4} + \dfrac{2}{x^3} f\left(\dfrac{y}{x}\right) + \dfrac{y}{x^4} f'\left(\dfrac{y}{x}\right)$

$\dfrac{\partial^2 F}{\partial y^2}(x,y) = \dfrac{1}{x^3} f''(y/x)$

$\dfrac{\partial^2 F}{\partial y \partial x}(x,y) = \dfrac{-2}{x^3} f'(y/x) - \dfrac{y}{x^4} \cdot f''(y/x)$

$$x^2 \frac{\partial^2 F}{\partial x^2}(x,y) = \cancel{f''\left(\frac{y}{x}\right) \cdot \frac{y^2}{x^3}} + \cancel{f'\left(\frac{y}{x}\right) \cdot \frac{3y}{x^2}} + \frac{2}{x} f(y/x) + \cancel{\frac{y}{x^2} f'\left(\frac{y}{x}\right)}$$

$$+ y^2 \frac{\partial^2 F}{\partial y^2}(x,y) = \cancel{\frac{y^2}{x^3} f''(y/x)}$$

$$+2xy \frac{\partial^2 F}{\partial y \partial x}(x,y) = \cancel{\frac{-4y}{x^2} f'(y/x)} \cancel{- \frac{2y^2}{x^3} \cdot f''(y/x)}$$

$$= \frac{2}{x} f(y/x) = \underline{\underline{2F(x,y)}}$$

9.12

$$u = \log\sqrt{(x-a)^2+(y-b)^2} = \frac{1}{2}\log\left[(x-a)^2+(y-b)^2\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\ \frac{2(x-a)}{(x-a)^2+(y-b)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\ \frac{2(y-b)}{(x-a)^2+(y-b)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\ \frac{2\left[(x-a)^2+(y-b)^2\right]-4(x-a)^2}{\left[(x-a)^2+(y-b)^2\right]^2} = \frac{(y-b)^2-(x-a)^2}{\left[(x-a)^2+(y-b)^2\right]^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\ \frac{2\left[(x-a)^2+(y-b)^2\right]-4(y-b)^2}{\left[(x-a)^2+(y-b)^2\right]^2} = \frac{(x-a)^2-(y-b)^2}{\left[(x-a)^2+(y-b)^2\right]^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

9.13

$u(x,y)$ satisfies $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$v(x,y) = u\underbrace{\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)}_{P}$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}(P)\cdot\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{\partial u}{\partial y}(P)\cdot\frac{-2xy}{(x^2+y^2)^2} =$$

$$= \frac{\partial u}{\partial x}(P)\cdot\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{\partial u}{\partial y}(P)\cdot\frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}(P)\cdot\frac{-2xy}{(x^2+y^2)^2} + \frac{\partial u}{\partial y}(P)\cdot\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}(P)\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)^2 + \frac{\partial^2 u}{\partial y\partial x}(P)\cdot\frac{-2xy\,(y^2-x^2)}{(x^2+y^2)^4} +$$

$$+ \frac{\partial u}{\partial x}(P)\cdot\frac{-2x\,(x^2+y^2)^2 - 2\,(x^2+y^2)\cdot 2x\,(y^2-x^2)}{(x^2+y^2)^4} +$$

$$+ \frac{\partial^2 u}{\partial x\partial y}(P)\cdot\frac{-2xy\,(y^2-x^2)}{(x^2+y^2)^4} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{4x^2y^2}{(x^2+y^2)^4} +$$

$$+ \frac{\partial u}{\partial y}(P)\cdot\frac{-2y\,(x^2+y^2)^2 + 2\,(x^2+y^2)\,2x\cdot 2xy}{(x^2+y^2)^4}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}(P) \cdot \frac{-2xy}{(x^2+y^2)^2} + \frac{\partial u}{\partial y}(P) \cdot \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}(P) \cdot \frac{4x^2y^2}{(x^2+y^2)^4} + \frac{\partial^2 u}{\partial x \partial y}(P) \cdot \frac{-2xy(x^2-y^2)}{(x^2-y^2)^4} +$$

$$+ \frac{\partial^2 u}{\partial x \partial y}(P) \cdot \frac{-2xy(x^2-y^2)}{(x^2-y^2)^4} + \frac{\partial^2 u}{\partial y^2}(P) \cdot \frac{(x^2-y^2)^2}{(x^2+y^2)^4}$$

$$+ \frac{\partial u}{\partial x}(P) \cdot \frac{-2x(x^2+y^2)^2 + 2(x^2+y^2) \cdot 2y \cdot 2xy}{(x^2+y^2)^4} +$$

$$+ \frac{\partial u}{\partial y}(P) \cdot \frac{-2y(x^2+y^2)^2 - 2(x^2+y^2)2y(x^2-y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial x^2}(P)+\frac{\partial^2 v}{\partial y^2}(P)=$$

$$\frac{\partial^2 v}{\partial x^2}\left\{\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}(P)\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)^2+\cancel{\frac{\partial^2 u}{\partial y\partial x}(P)\cdot\frac{-2xy(y^2-x^2)}{(x^2+y^2)^4}}+\\
&+\frac{\partial u}{\partial x}(P)\cdot\frac{-2x(x^2+y^2)^2-2(x^2+y^2)\cdot 2x(y^2-x^2)}{(x^2+y^2)^4}+\\
&+\cancel{\frac{\partial^2 u}{\partial x\partial y}(P)\cdot\frac{-2xy(y^2-x^2)}{(x^2+y^2)^4}}+\frac{\partial^2 u}{\partial y^2}\cdot\frac{4x^2y^2}{(x^2+y^2)^4}+\\
&+\frac{\partial u}{\partial y}(P)\cdot\frac{-2y(x^2+y^2)^2+2(x^2+y^2)2x\cdot 2xy}{(x^2+y^2)^4}+
\end{aligned}\right.$$

$$\frac{\partial^2 v}{\partial y^2}\left\{\begin{aligned}
&+\frac{\partial^2 u}{\partial x^2}(P)\cdot\frac{4x^2y^2}{(x^2+y^2)^4}+\cancel{\frac{\partial^2 u}{\partial x\partial y}(P)\cdot\frac{-2xy(x^2-y^2)}{(x^2-y^2)^4}}+\\
&+\cancel{\frac{\partial^2 u}{\partial x\partial y}(P)\cdot\frac{-2xy(x^2-y^2)}{(x^2-y^2)^4}}+\frac{\partial^2 u}{\partial y^2}(P)\cdot\frac{(x^2-y^2)^2}{(x^2+y^2)^4}\\
&+\frac{\partial u}{\partial x}(P)\cdot\frac{-2x(x^2+y^2)^2+2(x^2+y^2)\cdot 2y\cdot 2xy}{(x^2+y^2)^4}+\\
&+\frac{\partial u}{\partial y}(P)\cdot\frac{-2y(x^2+y^2)^2-2(x^2+y^2)2y(x^2-y^2)}{(x^2+y^2)^4}=
\end{aligned}\right.$$

$$=\frac{\partial^2 u}{\partial x^2}(P)\cancel{\frac{y^4+x^4+2x^2y^2}{(x^2+y^2)^4}}+\frac{\partial^2 u}{\partial y^2}(P)\cancel{\frac{x^4+y^4+2x^2y^2}{(x^2+y^2)^4}}+$$

$$+\frac{\partial u}{\partial x}(P)\frac{(x^2+y^2)\left[-2x(x^2+y^2)+8xy^2-2x(x^2+y^2)-4x(y^2-x^2)\right]}{(x^2+y^2)^4}$$

$$+\frac{\partial u}{\partial y}(P)\frac{(x^2+y^2)\left[-2y(x^2+y^2)-4y(x^2-y^2)-2y(x^2+y^2)+8x^2y\right]}{(x^2+y^2)^4}=$$

$$= \frac{\partial u}{\partial x}(P)\,\frac{-4x\,2y^2+8xy^2}{(x^2+y^2)^3} +$$

$$+ \frac{\partial u}{\partial y}(P)\,\frac{-4y\,2x^2+8x^2y}{(x^2+y^2)^4} = 0$$

9.14 $u = x - y + x^2 + 2xy + y^2 + x^3 - 3x^2y - y^3 + x^4 - 4x^2y^2 + y^4$

$$\frac{\partial u}{\partial x} = 1 + 2x + 2y + 3x^2 - 6xy + 4x^3 - 8xy^2$$

$$\frac{\partial u}{\partial y} = -1 + 2x + 2y - 3x^2 - 3y^2 - 8x^2y + 4y^3$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + 6x - 6y + 12x^2 - 8y^2$$

$$\frac{\partial^3 u}{\partial x^3} = 6 + 24x \qquad \frac{\partial^4 u}{\partial x^4} = 24$$

$$\boxed{\frac{\partial^4 u}{\partial x^3 \partial y} = 0}$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = -6 - 16y \qquad \boxed{\frac{\partial^4 u}{\partial x^2 \partial y^2} = -16}$$

9.17

$$\begin{cases} x^2+y-z^2-t^2=0 \\ x^2-y-z^2-t=0 \end{cases} \qquad (S)$$

¿(z,t) son funcions de (x,y) en un entorno del punto $(x,y)=(2,1)$ con $(z,t)=(1,2)$?

Definimos $g_1(x,y,z,t)=x^2+y-z^2-t^2$, $g_2(x,y,z,t)=x^2-y-z^2-t$,

$A=(2,1,1,2)$, $B=(2,1)$ y $C=(1,2)$. Ahora comprobamos:

(a) $g_1(2,1,1,2)=g_2(2,1,1,2)=0$

(b) $$\begin{vmatrix} \frac{\partial g_1}{\partial z}(A) & \frac{\partial g_1}{\partial t}(A) \\ \frac{\partial g_2}{\partial z}(A) & \frac{\partial g_2}{\partial t}(A) \end{vmatrix} = \begin{vmatrix} -2 & -4 \\ -2 & -1 \end{vmatrix} = 2-8=-6\neq 0$$

Así que aplicando el teorema de la función implícita, localmente z y t son funcions de x e y definidas por (S). Es decir, existe un entorno U de $(2,1)$, otro V de $(1,2)$ y una función única $(z,t): U \longrightarrow V$ tal que:

$$(x,y) \longrightarrow (z(x,y),t(x,y))$$

1. $g_i(x,y,z(x,y),t(x,y))=0 \quad i=1,2.$

2. $(z(B),t(B))=C$

Cálculo de las derivadas parciales

Ahora derivamos z y t parcialmente sabiendo que satisfacen:

$$\begin{cases} x^2+y-z^2-t^2=0 \\ x^2-y-z^2-t=0 \end{cases} \qquad (S)$$

Derivando respecto de x obtenemos:

$$(S1)\quad \left.\begin{aligned} &2x-2z\frac{\partial z}{\partial x}-2t\frac{\partial t}{\partial x}=0 \\ &2x-2z\frac{\partial z}{\partial x}-\frac{\partial t}{\partial x}=0 \end{aligned}\right\}$$

y particularizamos en $(x,y,z,t)=(2,1,1,2)$:

$$\begin{cases} 4-2\dfrac{\partial z}{\partial x}(B)-4\dfrac{\partial t}{\partial x}(B)=0 \\ 4-2\dfrac{\partial z}{\partial x}(B)-\dfrac{\partial t}{\partial x}(B)=0 \end{cases} \Longrightarrow -3\frac{\partial t}{\partial x}(B)=0$$

$$\Longrightarrow \frac{\partial t}{\partial x}(B)=0 \Longrightarrow \frac{\partial z}{\partial x}(B)=2$$

Derivando respecto de y obtenemos:

$$(S2)\quad \left.\begin{aligned} &1-2z\frac{\partial z}{\partial y}-2t\frac{\partial t}{\partial y}=0 \\ &-1-2z\frac{\partial z}{\partial y}-\frac{\partial t}{\partial y}=0 \end{aligned}\right\}$$

y particularizamos en $(x,y,z,t)=(2,1,1,2)$:

$$\begin{cases} 1-2\dfrac{\partial z}{\partial y}(B)-4\dfrac{\partial t}{\partial y}(B)=0 \\ -1-2\dfrac{\partial z}{\partial y}(B)-\dfrac{\partial t}{\partial y}(B)=0 \end{cases} \Longrightarrow 2-3\frac{\partial t}{\partial y}(B)=0$$

$$\Longrightarrow \frac{\partial t}{\partial y}(B)=\frac{2}{3} \Longrightarrow \frac{\partial z}{\partial y}(B)=\frac{-5}{6}$$

Derivando (S1) respecto de x tenemos:

$$\left.\begin{array}{l} 2-2\left(\dfrac{\partial z}{\partial x}\right)^2 - 2z\,\dfrac{\partial^2 z}{\partial x^2} - 2\left(\dfrac{\partial t}{\partial x}\right)^2 - 2t\,\dfrac{\partial^2 t}{\partial x^2} = 0 \\[2ex] 2-2\left(\dfrac{\partial z}{\partial x}\right)^2 - 2z\,\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 t}{\partial x^2} = 0 \end{array}\right\}$$

Particularizando en $(x,y,z,t)=(2,1,1,2)$ y teniendo en cuenta que $\dfrac{\partial z}{\partial x}(B)=2$ y $\dfrac{\partial t}{\partial x}=0$:

$$\left.\begin{array}{l} 2-8-2\,\dfrac{\partial^2 z}{\partial x^2} - 4\,\dfrac{\partial^2 t}{\partial x^2} = 0 \\[2ex] 2-8-2\,\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 t}{\partial x^2} = 0 \end{array}\right\} \Rightarrow \dfrac{\partial^2 t}{\partial x^2}(B)=0 \text{ y } \dfrac{\partial^2 z}{\partial x^2}(B)=-3$$

Derivando (S2) respecto de y tenemos:

$$\left.\begin{array}{l} -2\left(\dfrac{\partial z}{\partial y}\right)^2 - 2z\,\dfrac{\partial^2 z}{\partial y^2} - 2\left(\dfrac{\partial t}{\partial y}\right)^2 - 2t\,\dfrac{\partial^2 t}{\partial y^2} = 0 \\[2ex] -2\left(\dfrac{\partial z}{\partial y}\right)^2 - 2z\,\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial^2 t}{\partial y^2} = 0 \end{array}\right\}$$

Particularizando en $(x,y,z,t)=(2,1,1,2)$ y teniendo en cuenta que $\dfrac{\partial z}{\partial y}(B)=\dfrac{-5}{6}$ y $\dfrac{\partial t}{\partial y}=\dfrac{2}{3}$:

$$\left.\begin{array}{l} -\dfrac{25}{18} - 2\,\dfrac{\partial^2 z}{\partial y^2} - 4\,\dfrac{\partial^2 t}{\partial y^2} - \dfrac{8}{9} = 0 \\[2ex] -\dfrac{25}{18} - 2\,\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial^2 t}{\partial y^2} = 0 \end{array}\right\} \Rightarrow \dfrac{\partial^2 t}{\partial y^2}(B)=\dfrac{-8}{27} \text{ y } \dfrac{\partial^2 z}{\partial y^2}(B)=-\dfrac{59}{108}$$

Derivando (S1) respecto de y tenemos:

$$\left.\begin{array}{l} -2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} - 2z\dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial t}{\partial x}\dfrac{\partial t}{\partial y} - 2t\dfrac{\partial^2 t}{\partial x \partial y} = 0 \\ -2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} - 2z\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 t}{\partial x \partial y} = 0 \end{array}\right\}$$

Particularizando en $(x,y,z,t)=(2,1,1,2)$ y teniendo en cuenta que $\dfrac{\partial z}{\partial x}(B)=2$, $\dfrac{\partial t}{\partial x}=0$, $\dfrac{\partial t}{\partial y}=\dfrac{2}{3}$, $\dfrac{\partial z}{\partial y}=\dfrac{-5}{6}$:

$$\left.\begin{array}{l} \dfrac{10}{3} - 2\dfrac{\partial^2 z}{\partial x \partial y} - 4\dfrac{\partial^2 t}{\partial x \partial y} = 0 \\ \dfrac{10}{3} - 2\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 t}{\partial x \partial y} = 0 \end{array}\right\} \Rightarrow \dfrac{\partial^2 t}{\partial x \partial y}(B)=0 \,, \quad \dfrac{\partial^2 z}{\partial x \partial y}(B)=\dfrac{5}{3}$$

9.18

$$(S)\begin{cases} x e^t + yz - z^2 = 0 \\ y\cos t + x^2 - z^2 = 1 \end{cases} \qquad (x,y,z,t) = (2,1,2,0)$$

¿(z,t) son funcions de (x,y) en un entorno del punto $(x,y)=(2,1)$ con $(z,t)=(2,0)$?

Definimos $g_1(x,y,z,t) = x e^t + yz - z^2$, $g_2(x,y,z,t) = \cos t + x^2 - z^2 - 1$

$A=(2,1,2,0)$, $B=(2,1)$ y $C=(2,0)$. Ahora comprobamos:

ⓐ $g_1(2,1,2,0) = g_2(2,1,2,0) = 0$

ⓑ $$\begin{vmatrix} \frac{\partial g_1}{\partial z}(A) & \frac{\partial g_1}{\partial t}(A) \\ \frac{\partial g_2}{\partial z}(A) & \frac{\partial g_2}{\partial t}(A) \end{vmatrix} = \begin{vmatrix} y-2z & xe^t \\ -2z & -\operatorname{sen} t \end{vmatrix}_{(x,y,z,t)=A} = \begin{vmatrix} -3 & 2 \\ -4 & 0 \end{vmatrix} = 8 \neq 0$$

Así que aplicando el teorema de la función implícita, localmente z y t son funcions de x e y definidas por (S). Es decir, existe un entorno U de $(2,1)$, otro V de $(2,0)$ y una función única $(z,t): U \longrightarrow V$, $(x,y) \longrightarrow (z(x,y), t(x,y))$ tal que:

1. $g_i(x,y,z(x,y),t(x,y)) = 0 \quad i=1,2.$

2. $(z(B), t(B)) = C$

Cálculo de las derivadas parciales

Ahora derivamos z y t parcialmente sabiendo que satisfacen:

$$(S)\begin{cases} x e^{t} + yz - z^2 = 0 \\ y\cos t + x^2 - z^2 = 1 \end{cases}$$

Derivando respecto de x obtenemos:

$$(S1)\left.\begin{array}{l} e^{t} + x e^{t}\dfrac{\partial t}{\partial x} + y\dfrac{\partial z}{\partial x} - 2z\dfrac{\partial z}{\partial x} = 0 \\ -y\,\mathrm{sen}(t)\dfrac{\partial t}{\partial x} + 2x - 2z\dfrac{\partial z}{\partial x} = 0 \end{array}\right\}$$

y particularizamos en $(x,y,z,t) = (2,1,2,0)$:

$$\begin{cases} 1 + 2\dfrac{\partial t}{\partial x} - 3\dfrac{\partial z}{\partial x} = 0 \\ -4 - 4\dfrac{\partial z}{\partial x} = 0 \end{cases} \Rightarrow \frac{\partial z}{\partial x}(B) = \frac{\partial t}{\partial x}(B) = 1$$

$$(S)\begin{cases} x e^{t} + yz - z^2 = 0 \\ y\cos t + x^2 - z^2 = 1 \end{cases}$$

Derivando respecto de y obtenemos:

$$(S2)\left.\begin{array}{l} x e^{t}\dfrac{\partial t}{\partial y} + z + y\dfrac{\partial z}{\partial y} - 2z\dfrac{\partial z}{\partial y} = 0 \\ \cos t - y\,\mathrm{sen}(t)\dfrac{\partial t}{\partial y} - 2z\dfrac{\partial z}{\partial y} = 0 \end{array}\right\}$$

y particularizamos en $(x,y,z,t) = (2,1,2,0)$:

$$\begin{cases} 2\dfrac{\partial t}{\partial y} + 2 - 3\dfrac{\partial z}{\partial y} = 0 \\ 1 - 4\dfrac{\partial z}{\partial y} = 0 \end{cases} \Rightarrow \frac{\partial z}{\partial y}(B) = \frac{1}{4}\,;\quad \frac{\partial t}{\partial y}(B) = \frac{-5}{8}$$

Derivando (S1) respecto de x tenemos:

$$\left.\begin{aligned}&e^t\frac{\partial t}{\partial x}+e^t\frac{\partial t}{\partial x}+xe^t\left(\frac{\partial t}{\partial x}\right)^2+xe^t\frac{\partial^2 t}{\partial x^2}+y\frac{\partial^2 z}{\partial x^2}-2\left(\frac{\partial z}{\partial x}\right)^2-2z\frac{\partial^2 z}{\partial x^2}=0\\&-y\cos(t)\left(\frac{\partial t}{\partial x}\right)^2-y\,\mathrm{sen}\,t\,\frac{\partial^2 t}{\partial x^2}+2-2\left(\frac{\partial z}{\partial x}\right)^2-2z\frac{\partial^2 z}{\partial x^2}=0\end{aligned}\right\}$$

Particularizando en $(x,y,z,t)=(2,1,2,0)$ y teniendo en cuenta que $\frac{\partial z}{\partial x}(B)=1$ y $\frac{\partial t}{\partial x}=1$:

$$\left.\begin{aligned}&2+2+2\frac{\partial^2 t}{\partial x^2}-2-3\frac{\partial^2 z}{\partial x^2}=0\\&-1-2+2-4\frac{\partial^2 z}{\partial x^2}=0\end{aligned}\right\}\Rightarrow\frac{\partial^2 z}{\partial x^2}(B)=\frac{-1}{4}\text{ y }\frac{\partial^2 t}{\partial x^2}(B)=\frac{-11}{8}$$

Derivando (S2) respecto de y tenemos:

$$\left.\begin{aligned}&xe^t\left(\frac{\partial t}{\partial y}\right)^2+xe^t\frac{\partial^2 t}{\partial y^2}+\frac{\partial z}{\partial y}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}-2\left(\frac{\partial z}{\partial y}\right)^2-2z\frac{\partial^2 z}{\partial y^2}=0\\&-\mathrm{sen}\,t\,\frac{\partial t}{\partial y}-\mathrm{sen}\,t\,\frac{\partial t}{\partial y}-y\cos t\left(\frac{\partial t}{\partial y}\right)^2-y\,\mathrm{sen}\,t\,\frac{\partial^2 t}{\partial y^2}-2\left(\frac{\partial z}{\partial y}\right)^2-2z\frac{\partial^2 z}{\partial y^2}=0\end{aligned}\right\}$$

Particularizando en $(x,y,z,t)=(2,1,2,0)$ y teniendo en cuenta que $\frac{\partial z}{\partial y}(B)=\frac{+1}{4}$ y $\frac{\partial t}{\partial y}=\frac{-5}{8}$:

$$\left.\begin{aligned}&\frac{50}{64}+2\frac{\partial^2 t}{\partial y^2}+\frac{2}{4}+\frac{\partial^2 z}{\partial y^2}-2\frac{1}{16}-4\frac{\partial^2 z}{\partial y^2}=0\\&-\frac{25}{64}-2\frac{1}{16}-4\frac{\partial^2 z}{\partial y^2}=0\end{aligned}\right\}\Rightarrow\frac{\partial^2 t}{\partial y^2}(B)=\frac{-395}{512},\ \frac{\partial^2 z}{\partial y^2}(B)=\frac{-33}{256}$$

Derivando (S1) respecto de y tenemos:

$$e^t \frac{\partial t}{\partial y} + x e^t \frac{\partial t}{\partial y} \frac{\partial t}{\partial y} + x e^t \frac{\partial^2 t}{\partial x \partial y} + \frac{\partial z}{\partial x} + y \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - 2z \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$-\operatorname{sen} t \frac{\partial t}{\partial x} - y \cos(t) \frac{\partial t}{\partial x} \frac{\partial t}{\partial y} - y \operatorname{sen} t \frac{\partial^2 t}{\partial x \partial y} - 2 \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} - 2z \frac{\partial^2 z}{\partial x \partial y} = 0$$

Particularizando en $(x,y,z,t) = (2,1,2,0)$ y teniendo en cuenta que $\frac{\partial z}{\partial x}(B) = 1$, $\frac{\partial t}{\partial x} = 1$, $\frac{\partial z}{\partial y} = \frac{1}{4}$ y $\frac{\partial t}{\partial y} = \frac{-5}{8}$:

$$\left.\begin{array}{l} -\frac{5}{8} - \frac{5}{4} + 2 \frac{\partial^2 t}{\partial x \partial y} + 1 - 3 \frac{\partial^2 z}{\partial x \partial y} - \frac{1}{2} = 0 \\ +\frac{5}{8} - 2 - 4 \frac{\partial^2 z}{\partial x \partial y} = 0 \end{array}\right\} \Rightarrow$$

$$\frac{\partial^2 z}{\partial x \partial y}(2,1) = \frac{1}{4}\left(\frac{5}{8} - \frac{16}{8}\right) = \frac{-11}{32}$$

$$\frac{\partial^2 t}{\partial x \partial y}(2,1) = \frac{1}{2}\left[-\frac{1}{2} + 3 \frac{-11}{32} + \frac{5}{4} + \frac{5}{8}\right] = \frac{11}{64}$$

9.19

$$(S)\begin{cases} x^2+y^2+z^2=20 \\ x-xy+z=4 \end{cases} \qquad (x,y,z)=(0,2,4)$$

1ª Parte (y,z) son funcions de x en un entorno del punto $x=0$ con $(y,z)=(2,4)$?

Definimos $g_1(x,y,z)=x^2+y^2+z^2-20$, $g_2(x,y,z)=x-xy+z-4$

$A=(0,2,4)$, $B=0$ y $C=(2,4)$. Ahora comprobamos:

ⓐ $g_1(0,2,4)=g_2(0,2,4)=0$

ⓑ
$$\begin{vmatrix} \frac{\partial g_1}{\partial y}(A) & \frac{\partial g_1}{\partial z}(A) \\ \frac{\partial g_2}{\partial y}(A) & \frac{\partial g_2}{\partial z}(A) \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ -x & 1 \end{vmatrix}_{(x,y,z)=A} = \begin{vmatrix} 4 & 8 \\ 0 & 1 \end{vmatrix} = 4 \neq 0$$

Así que aplicando el teorema de la función implícita, localmente y y z son funcions de x definidas por (S). Es decir, existe un entorno U de 0, otro V de $(2,4)$ y una función única $(y,z): U \longrightarrow V$ tal que:

$$x \longrightarrow (y(x), z(x))$$

1. $g_i(x, y(x), z(x)) = 0 \qquad i=1,2.$

2. $(y(0), z(0)) = (2,4)$

Cálculo de las derivadas

Ahora derivamos z e y sabiendo que satisfacen:

$$(S)\begin{cases} x^2+y^2+z^2=20 \\ x-xy+z=4 \end{cases}$$

Derivando respecto de x obtenemos:

$$(S1)\quad \left.\begin{aligned} 2x+2y\,y'+2z\,z'&=0 \\ 1-y-xy'+z'&=0 \end{aligned}\right\}$$ y particularizamos en $(x,y,z)=(0,2,4)$:

$$\begin{cases} 4y'(0)+8z'(0)=0 \\ 1-2+z'(0)=0 \end{cases} \Rightarrow \quad \underline{z'(0)=1}, \quad \underline{y'(0)=-2}$$

Derivando (S1) respecto de x obtenemos:

$$(S2)\quad \left.\begin{aligned} 2+2(y')^2+2y\,y''+2(z')^2+2z\,z''&=0 \\ -y'-y'-xy''+z''&=0 \end{aligned}\right\}$$ y particularizando en $(x,y,z)=(0,2,4)$:

$$\left.\begin{aligned} 2+8+4y''(0)+2+8z''(0)=0 \\ 4+z''(0)=0 \end{aligned}\right\} \Rightarrow z''(0)=-4, \quad y''(0)=-5$$

2ª Parte (x,z) son funcions de y en un entorno del punto $y=2$ con $(x,z)=(0,4)$?

Definimos $g_1(x,y,z)=x^2+y^2+z^2-20$, $g_2(x,y,z)=x-xy+z-4$

$A=(0,2,4)$. Comprobamos:

ⓐ $g_1(0,2,4)=g_2(0,2,4)=0$

ⓑ $$\begin{vmatrix} \frac{\partial g_1}{\partial x}(A) & \frac{\partial g_1}{\partial z}(A) \\ \frac{\partial g_2}{\partial x}(A) & \frac{\partial g_2}{\partial z}(A) \end{vmatrix} = \begin{vmatrix} 2x & 2z \\ 1-y & 1 \end{vmatrix}_{(x,y,z)=A} = \begin{vmatrix} 0 & 8 \\ -1 & 1 \end{vmatrix} = 8 \neq 0$$

Así que aplicando el teorema de la función implícita, localmente x y z son funcions de y definidas por (S). Es decir, existe un entorno U de 2, otro V de $(0,4)$ y una función única $(x,z): U \longrightarrow V$ tal que:

$$y \longrightarrow (x(y), z(y))$$

1. $g_i(x(y), y, z(y)) = 0$ $\quad i=1,2.$

2. $(x(2), z(2)) = (0,4)$

Cálculo de las derivadas

Ahora derivamos x y z sabiendo que satisfacen:

$$(S)\begin{cases} x^2+y^2+z^2=20 \\ x-xy+z=4 \end{cases}$$

Derivando respecto de y obtenemos:

$$(S1)\quad \left.\begin{aligned} &2xx'+2y+2zz'=0 \\ &x'-x'y-x+z'=0 \end{aligned}\right\}$$

y particularizamos en $(x,y,z) = (0,2,4)$:

$$\begin{cases} 4+8z'(2)=0 \\ x'(2)-2x'(2)+z'(2)=0 \end{cases} \Rightarrow \quad \underline{z'(2)=\frac{-1}{2}}, \quad \underline{x'(2)=-\frac{1}{2}}$$

Derivando (S1) respecto de y obtenemos:

$$(S2)\quad \left.\begin{aligned} &2(x')^2+2xx''+2+2(z')^2+2zz''=0 \\ &x''-x''y-x'-x'+z''=0 \end{aligned}\right\}$$

y particularizando en $(x,y,z)=(0,2,4)$:

$$\left.\begin{aligned} &\frac{1}{2}+2+\frac{1}{2}+8z''(2)=0 \\ &x''(2)-2x''(2)+1+z''(2)=0 \end{aligned}\right\} \Rightarrow z''(2)=\frac{-3}{8} \text{ y } x''(2)=5/8$$

3ª Parte ¿(x,y) son funciones de z en un entorno del punto $z=4$ con $(x,y)=(0,2)$?

Definimos $g_1(x,y,z) = x^2+y^2+z^2-20$, $g_2(x,y,z) = x-xy+z-4$

$A=(0,2,4)$. Comprobamos:

ⓐ $g_1(0,2,4) = g_2(0,2,4) = 0$

ⓑ $$\begin{vmatrix} \frac{\partial g_1}{\partial x}(A) & \frac{\partial g_1}{\partial y}(A) \\ \frac{\partial g_2}{\partial x}(A) & \frac{\partial g_2}{\partial y}(A) \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 1-y & -x \end{vmatrix}_{(x,y,z)=A} = \begin{vmatrix} 0 & 4 \\ -1 & 0 \end{vmatrix} = 4 \neq 0$$

Así que aplicando el teorema de la función implícita, localmente x e y son funciones de z definidas por (S). Es decir, existe un entorno U de 4, otro V de $(0,2)$ y una función única $(x,y): U \longrightarrow V$ tal que:

$$z \longrightarrow (x(z), y(z))$$

1. $g_i(x(z), y(z), z) = 0$ $\quad i=1,2.$

2. $(x(4), y(4)) = (0,2)$

Cálculo de las derivadas

Ahora derivamos x e y sabiendo que satisfacen:

$$(S)\begin{cases} x^2+y^2+z^2=20 \\ x-xy+z=4 \end{cases}$$

Derivando respecto de z obtenemos:

$$(S1)\left.\begin{matrix} 2xx'+2yy'+2z=0 \\ x'-x'y-xy'+1=0 \end{matrix}\right\}$$ y particularizamos en $(x,y,z) = (0,2,4)$:

$$\left\{\begin{matrix} 4y'(4)+8=0 \\ -x'(4)+1=0 \end{matrix}\right. \Rightarrow \underline{x'(4)=1}, \quad \underline{y'(4)=-2}$$

Derivando (S1) respecto de z obtenemos:

$$(S2)\left.\begin{matrix} 2(x')^2+2xx''+2(y')^2+2yy''+2=0 \\ x''-x''y-x'y'-x'y'-xy''=0 \end{matrix}\right\}$$ y particularizando en $(x,y,z)=(0,2,4)$:

$$\left.\begin{matrix} 2+8+4y''(4)+2=0 \\ -x''(4)+4=0 \end{matrix}\right\} \Rightarrow x''(4)=4, \quad y''(4)=-3$$

9.22

$$f(x,y) = \left(\frac{x^2}{1-x^2-y^2} , \frac{y^2}{1-x^2-y^2} \right)$$

$$\frac{\partial f_1}{\partial x}(x,y) = \frac{2x(1-x^2-y^2)+2x\cdot x^2}{(1-x^2-y^2)^2} = \frac{-2xy^2+2x}{(1-x^2-y^2)^2}$$

$$\frac{\partial f_1}{\partial y}(x,y) = \frac{-2y\,x^2}{(1-x^2-y^2)^2}$$

$$\frac{\partial f_2}{\partial x}(x,y) = \frac{-2xy^2}{(1-x^2-y^2)^2}$$

$$\frac{\partial f_2}{\partial y}(x,y) = \frac{2y(1-x^2-y^2)+2y\cdot y^2}{(1-x^2-y^2)^2} = \frac{2y-2yx^2}{(1-x^2-y^2)^2}$$

$$Jf(x,y) = \frac{2}{(1-x^2-y^2)^2} \begin{pmatrix} x-xy^2 & -yx^2 \\ -xy^2 & y-yx^2 \end{pmatrix} =$$

$$|Jf(x,y)| = \frac{4xy}{(1-x^2-y^2)^4} \begin{vmatrix} 1-y^2 & -yx \\ -xy & 1-x^2 \end{vmatrix} =$$

$$= \frac{4xy}{(1-x^2-y^2)^4} \cdot \left[(1-x^2)(1-y^2) + x^2y^2\right] = \frac{4xy}{(1-x^2-y^2)^4} \cdot \left[1-x^2-y^2+2x^2y^2\right]$$

$$= 0 \iff \begin{cases} xy=0 < \begin{matrix} x=0 \\ \text{ó} \\ y=0 \end{matrix} \\ \text{ó} \\ 1 = x^2+y^2-2x^2y^2 \end{cases}$$

Si (x,y) es tal que $x, y \neq 0$ y $1 \neq x^2+y^2-2x^2y^2$ entonces f es localmente invertible en este punto.

9.23 b) $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$ $x>0$, $y>0$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}(x,y) = y - \frac{50}{x^2} = 0 \\ \frac{\partial f}{\partial y}(x,y) = x - \frac{20}{y^2} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y = \frac{50}{x^2} \\ x = \frac{20}{y^2} \end{array}\right\} \Rightarrow$$

$$\Rightarrow y = \frac{50}{\frac{20^2}{y^4}} \Rightarrow \frac{20^2}{y^3} = 50 \Rightarrow 40 = 5y^3 \Rightarrow 8 = y^3$$

$$\Rightarrow y = 2 \quad , x = 5$$

El único punto candidato a extremo es el $(5,2)$

Calculemos la matriz hessiana:

$$\frac{\partial^2 f}{\partial x^2} = 100x^{-3} \qquad \frac{\partial^2 f}{\partial y^2} = +40y^{-3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

$$Hf(5,2) = \begin{pmatrix} 100\frac{1}{\sqrt[3]{5}} & 1 \\ 1 & 40\frac{1}{\sqrt[3]{2}} \end{pmatrix}$$

$$1, \Delta_1, \Delta_2 \longrightarrow 1, \underbrace{100\frac{1}{\sqrt[3]{5}}}_{>0}, \underbrace{4000\frac{1}{\sqrt[3]{10}} - 1}_{>0} \Rightarrow$$

$(5,2)$ es un mínimo relativo de f.

9.23.e) $f(x,y,z) = \operatorname{sen} x + \operatorname{sen} y + \operatorname{sen} z - \operatorname{sen}(x+y+z)$ $\quad x,y,z \in (0,\pi)$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \cos x - \cos(x+y+z) = 0 \\ \frac{\partial f}{\partial y} &= \cos y - \cos(x+y+z) = 0 \\ \frac{\partial f}{\partial z} &= \cos z - \cos(x+y+z) = 0 \end{aligned}\right\} \Rightarrow \cos x = \cos y = \cos z \Rightarrow$$

$x = y = z$ (la función coseno es inyectiva en $(0,\pi)$)

Además se debe verificar que $\cos x - \cos(3x) = 0 \Rightarrow$

$\cos(x) = \cos(3x) = \cos(2x+x) = \cos(2x)\cos x - \operatorname{sen}(2x)\operatorname{sen} x$

Analizamos esta ecuación. A Si $x \in (0, \pi/2]$ entonces $3x \in \left(0, \frac{3\pi}{2}\right]$ y $\cos x \in [0,1)$. Además:

* Si $x \in \left(0, \frac{\pi}{6}\right]$, $3x \in \left(0, \frac{\pi}{2}\right]$ y $\cos(x) \neq \cos(3x)$ porque $x \neq 3x$ y $\cos x$ es inyectiva en $\left(0, \frac{\pi}{2}\right]$

* Si $x \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right]$, $3x \in \left(\frac{\pi}{2}, \pi\right]$ y $\cos x > 0$ mientras que $\cos 3x < 0 \Rightarrow \cos x \neq \cos 3x$

* Si $x \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $3x \in \left(\pi, \frac{3\pi}{2}\right)$ y $\cos x > 0$ mientras que $\cos 3x < 0 \Rightarrow \cos x \neq \cos 3x$.

* Si $x = \frac{\pi}{2}$ y $3x = \frac{3\pi}{2}$ entonces $\cos x = 0 = \cos 3x$

$$\frac{5\pi}{6} + \frac{\pi}{6} = \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

(B) Si $x \in \left(\frac{\pi}{2}, \pi\right)$ entonces $3x \in \left(\frac{3\pi}{2}, 3\pi\right]$ y $\cos x \in (-1, 0)$. Analizamos la ecuacion en subintervalos:

* Si $x \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$, $3x \in \left(\frac{3\pi}{2}, 2\pi\right]$ y $\cos(x) \neq \cos(3x)$ porque $\cos(x) < 0$ y $\cos(3x) > 0$

* Si $x \in \left(\frac{2\pi}{3}, \frac{5\pi}{6}\right]$, $3x \in \left(2\pi, \frac{5\pi}{2}\right]$ y $\cos x < 0$ mientras que $\cos 3x \geqslant 0 \Rightarrow \cos x \neq \cos 3x$

* Si $x \in \left(\frac{5\pi}{6}, \pi\right)$, $3x \in \left(\frac{5\pi}{2}, 3\pi\right)$ y $\cos x = \cos 3x$ entonces

como x y $3x$ están en el segundo cuadrante $3x = x + 2\pi \Rightarrow$
$\Rightarrow x = \pi$, pero esta solución no vale porque $x \in (0, \pi)$

Así que el único punto candidato a extremo relativo es

$$P = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

Calculamos la matriz hessiana:

$$\frac{\partial^2 f}{\partial x^2} = -\operatorname{sen} x + \operatorname{sen}(x+y+z) \qquad \frac{\partial^2 f}{\partial x \partial y} = \operatorname{sen}(x+y+z) \qquad \frac{\partial^2 f}{\partial x \partial z} = \operatorname{sen}(x+y+z)$$

$$\frac{\partial^2 f}{\partial y^2} = -\operatorname{sen} y + \operatorname{sen}(x+y+z) \qquad \frac{\partial^2 f}{\partial y \partial z} = \operatorname{sen}(x+y+z)$$

$$\frac{\partial^2 f}{\partial z^2} = -\operatorname{sen} z + \operatorname{sen}(x+y+z)$$

$$Hf\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} \longrightarrow \overset{}{1}, \overset{\Delta_1}{-2}, \overset{\Delta_2}{3}, \overset{\Delta_3}{-4} \Rightarrow$$

$\Rightarrow$ P es un máximo relativo.

$\boxed{9.24.b}$ $f(x,y) = x^2+y^2$ $\qquad \frac{x}{a}+\frac{y}{b}=1$

$L(x,y) = x^2+y^2+\lambda\left(\frac{x}{a}+\frac{y}{b}-1\right)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x+\frac{\lambda}{a} = 0 \\ \frac{\partial L}{\partial y} &= 2y+\frac{\lambda}{b}=0 \\ \frac{x}{a}+\frac{y}{b} &=1 \end{aligned}\right\} \begin{aligned} &\Rightarrow \lambda = -2ax = -2by \Rightarrow y = \frac{a}{b}x \\ &\Rightarrow \frac{x}{a}+\frac{a}{b}x\frac{1}{b} = 1 \Rightarrow \\ &\Rightarrow x\cdot\left(\frac{1}{a}+\frac{a}{b^2}\right)=1 \Rightarrow x = \frac{ab^2}{b^2+a^2} \end{aligned}$$

$\Rightarrow y = \frac{a^2b}{b^2+a^2}$. El único candidato a extremo es $P = \left(\frac{ab^2}{a^2+b^2}, \frac{ba^2}{a^2+b^2}\right)$

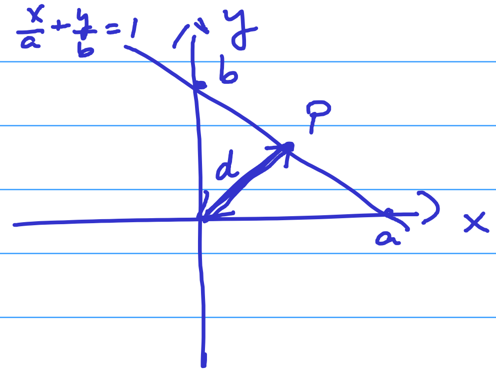
con $\lambda = -2\frac{a^2b^2}{a^2+b^2}$

$\frac{\partial^2 L}{\partial x^2} = 2 \qquad \frac{\partial^2 L}{\partial x\partial y} = 0 \qquad \frac{\partial^2 L}{\partial y^2} = 2$

$$HL(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$Q = (h_1\, h_2)\; HL(x,y)\begin{pmatrix} h_1 \\ h_2\end{pmatrix} = 2(h_1^2+h_2^2) > 0$ si $(h_1,h_2)\neq(0,0)$ por lo que P es un mínimo relativo condicionado

Dibujo



9.24d

$f(x,y,z) = \operatorname{sen} x \operatorname{sen} y \operatorname{sen} z \qquad x+y+z = \frac{\pi}{2} \qquad x,y,z \in \mathbb{R}^+$

$L(x,y,z) = \operatorname{sen} x \operatorname{sen} y \operatorname{sen} z + \lambda\left(x+y+z-\frac{\pi}{2}\right)$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x} = \cos x \operatorname{sen} y \operatorname{sen} z + \lambda = 0 \\ \dfrac{\partial L}{\partial y} = \operatorname{sen} x \cos y \operatorname{sen} z + \lambda = 0 \\ \dfrac{\partial L}{\partial z} = \operatorname{sen} x \operatorname{sen} y \cos z + \lambda = 0 \end{array}\right\} \Rightarrow \begin{cases} \cos x \operatorname{sen} y \operatorname{sen} z = \operatorname{sen} x \cos y \operatorname{sen} z \\ \cos x \operatorname{sen} y \operatorname{sen} z = \operatorname{sen} x \operatorname{sen} y \cos z \end{cases}$$

① Si $\operatorname{sen} z = 0$ entonces $\operatorname{sen} x \operatorname{sen} y = 0 \Rightarrow \operatorname{sen} x = 0$ ó $\operatorname{sen} y = 0$

$\operatorname{sen} z = 0 \Leftrightarrow z = 0$, $\quad \operatorname{sen} x = 0 \Leftrightarrow x = 0$, $\quad \operatorname{sen} y = 0 \Leftrightarrow y = 0$

$P_1 = \left(0, \frac{\pi}{2}, 0\right) \qquad P_2 = \left(\frac{\pi}{2}, 0, 0\right)$

② Si $\operatorname{sen} z \neq 0$ entonces $\begin{cases} \cos x \operatorname{sen} y = \operatorname{sen} x \cos y \\ \cos x \operatorname{sen} y \operatorname{sen} z = \operatorname{sen} x \operatorname{sen} y \cos z \end{cases}$

$\Rightarrow \operatorname{sen} x \cos y \operatorname{sen} z = \operatorname{sen} x \operatorname{sen} y \cos z$

2.1 Si $\operatorname{sen} x = 0 \Rightarrow \operatorname{sen} y = 0 \Rightarrow x = y = 0,\ z = \pi/2$

$P_3 = (0, 0, \pi/2)$

2.2 Si $\operatorname{sen} x \neq 0 \Rightarrow \cos y \operatorname{sen} z = \operatorname{sen} y \cos z$.

2.2 a $\quad \cos z = 0 \Rightarrow \cos y = 0 \Rightarrow z = y = \frac{\pi}{2}$ (IMPOSIBLE)

2.2.b $\quad \cos y = 0 \Rightarrow \cos z = 0 \Rightarrow z = y = \frac{\pi}{2}$

2.2.c $\quad \cos z \neq 0 \neq \cos y \Rightarrow \operatorname{tg} y = \operatorname{tg} z \Rightarrow y = z$

$$\cos x \operatorname{sen} y = \cos y \operatorname{sen} x \underset{\Downarrow}{\Rightarrow} \operatorname{tg} y = \operatorname{tg} x \Rightarrow y = z = x$$

(se razona como en 2.2.a y 2.2.b)

Así que $y = z = x = \frac{\pi}{6}$ $\qquad P_4 = \left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right)$

Resumen de puntos candidatos a extremos

$P_1 = \left(0, \frac{\pi}{2}, 0\right) \qquad \lambda = 0$

$P_2 = \left(\frac{\pi}{2}, 0, 0\right) \qquad \lambda = 0$

$P_3 = \left(0, 0, \pi/2\right) \qquad \lambda = 0$

$P_4 = \left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right) \qquad \lambda = -\cos\frac{\pi}{6}\left(\operatorname{sen}\frac{\pi}{6}\right)^2 = -\frac{\sqrt{3}}{2}\cdot\frac{1}{4}$

Calculamos la matriz hessiana:

$$\frac{\partial^2 L}{\partial x^2} = -\operatorname{sen} x \operatorname{sen} y \operatorname{sen} z = \frac{\partial^2 L}{\partial y^2} = \frac{\partial^2 L}{\partial z^2}$$

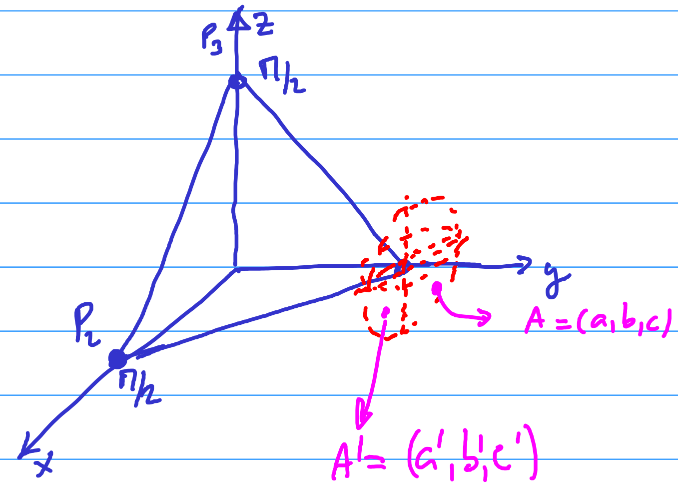
$$\frac{\partial^2 L}{\partial x \partial y} = \cos x \cos y \operatorname{sen} z \qquad \frac{\partial^2 L}{\partial x \partial z} = \cos x \operatorname{sen} y \cos z$$

$$\frac{\partial^2 L}{\partial y \partial z} = \operatorname{sen} x \cos y \cos z$$

Estudio en P_1 $\qquad HL(x,y) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\left.\begin{array}{l} (h_1, h_2, h_3)\ HL(x,y,z) \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 2h_1h_3 = Q \\ \\ Dg(P_1)(h_1, h_2, h_3) = h_1 + h_2 + h_3 = 0 \end{array}\right\} \begin{array}{l} Q \text{ puede anularse} \\ (h_3 = 0,\ h_1 = 1,\ h_2 = -1) \end{array}$$

Así que el método no asegura nada sobre P_1. No obstante $f(P_1) = 0$



$$\left.\begin{array}{l} a < 0 \\ b > 0 \\ c < 0 \end{array}\right\} f(a,b,c) > 0$$

$$\left.\begin{array}{l} a' > 0 \\ b' > 0 \\ c' < 0 \end{array}\right\} f(a', b', c') < 0$$

Así que P_1 no puede ser un extremo relativo condicionado y con un razonamiento similar se demuestra que P_2 y P_3 tampoco son extremos.

Estudio de P_4

$$HL\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right) = \begin{pmatrix} 1/8 & 3/8 & 3/8 \\ 3/8 & 1/8 & 3/8 \\ 3/8 & 3/8 & 1/8 \end{pmatrix}$$

$$(h_1\ h_2\ h_3)\ HL\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \frac{1}{8}\Big[\underbrace{h_1^2 + h_2^2 + h_3^2 + 6h_1h_2 + 6h_1h_3 + 6h_2h_3}_{Q}\Big]$$

(h_1, h_2, h_3) verifica $\quad Dg(P_4)(h_1, h_2, h_3)^t = h_1 + h_2 + h_3 = 0$

Así que $Q = \frac{1}{8}\left\{\left[h_1^2 + h_2^2 + (h_2+h_1)^2\right] + 6h_1h_2 - 6h_1(h_1+h_2) - 6h_2(h_1+h_2)\right\}$

$= \frac{1}{8}\left[2h_1^2 + 2h_2^2 - 2h_1h_2 + 6h_1h_2 - 6h_1^2 - 6h_1h_2 - 6h_1h_2 - 6h_2^2\right] =$

$= \frac{1}{8}\left[-4h_1^2 - 4h_2^2 - 8h_1h_2\right] = \frac{-4}{8}\left[h_1^2 + h_2^2 + 2h_1h_2\right] =$

$= \frac{-4}{8}(h_1+h_2)^2 < 0$

Así que P_4 es un máximo relativo condicionado.

9.24 e $f(x,y,z)=xy+yz \qquad x^2+y^2=2 \qquad y+z=2$

$$L(x,y,z) = xy+yz+\lambda(x^2+y^2-2)+\mu(y+z-2)$$

$$\frac{\partial L}{\partial x} = y+2\lambda x = 0 \qquad\qquad y=-2\lambda x=-\mu$$

$$\frac{\partial L}{\partial y} = x+z+2\lambda y+\mu=0 \qquad\qquad x+2+2\lambda x-4\lambda^2 x+2\lambda x=0$$

$$\frac{\partial L}{\partial z} = y+\mu=0$$

$$y+z=2 \qquad\qquad z=2-y$$

$$x^2+y^2=2$$

$$\Rightarrow x(1+4\lambda-4\lambda^2)+2=0 \Rightarrow \quad x=\frac{-2}{1+4\lambda-4\lambda^2}$$

$$y=\frac{+4\lambda}{1+4\lambda-4\lambda^2}$$

$$z=2-\frac{4\lambda}{1+4\lambda-4\lambda^2}$$

$$x^2+y^2=\frac{4+16\lambda^2}{(1+4\lambda-4\lambda^2)^2}=2 \Rightarrow \quad 4+16\lambda^2=2+32\lambda^2+32\lambda^4+16\lambda-16\lambda^2-64\lambda^3$$

$$\Rightarrow 32\lambda^4-64\lambda^3+16\lambda-2=0 \Rightarrow 16\lambda^4-32\lambda^3+8\lambda-1=0$$

$$\begin{array}{r|rrrrr} & 16 & -32 & 0 & 8 & -1 \\ -1 & & -16 & -16 & -16 & -8 \\ \hline & 16 & -16 & -16 & -8 & \end{array}$$

9.25 $f(x,y) = x^2+y^2$ $\qquad$ $4x^2+y^2 = 4$

En este problema no es necesario usar el método de los multiplicadores de Lagrange ya que se puede sustituir la ligadura en la función:

$$y^2 = 4(1-x^2)$$

$$f(x,y) \rightsquigarrow f(x) = x^2+4(1-x^2) = 4-3x^2$$

$$f'(x) = -6x = 0 \Rightarrow x=0$$

$$f''(x) = -6 < 0 \Rightarrow$$

$f(x,y)$ tiene máximos relativos condicionados en los puntos $(0,2)$ y $(0,-2)$. Sin embargo por estar f definida en un compacto debería alcanzar un mínimo absoluto. ¿Por qué no hemos localizado dicho mínimo?

USANDO LOS MULTIPLICADORES DE LAGRANGE

$$L(x,y) = x^2+y^2+\lambda(4x^2+y^2-4)$$

$$\frac{\partial L}{\partial x} = 2x+8\lambda x = 0 \Rightarrow 2x(1+4\lambda)=0 \quad \begin{cases} x=0 \\ \lambda = \frac{-1}{4} \end{cases}$$

$$\frac{\partial L}{\partial y} = 2y+2\lambda y = 0 \Rightarrow 2y(1+\lambda)=0 \quad \begin{cases} y=0 \\ \lambda=-1 \end{cases}$$

$$\underbrace{4x^2+y^2-4}_{g(x,y)}=0$$

Las soluciones del sistema anterior son:

P_1 $(0, 2)$ $\lambda = -1$
P_2 $(0, -2)$ $\lambda = -1$
P_3 $(1, 0)$ $\lambda = -1/4$
P_4 $(-1, 0)$ $\lambda = -1/4$

$$\frac{\partial^2 L}{\partial x^2} = 2 + 8\lambda \qquad HL(x,y) = \begin{pmatrix} 2+8\lambda & 0 \\ 0 & 2+2\lambda \end{pmatrix}$$

$$\frac{\partial^2 L}{\partial y^2} = 2 + 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y} = 0$$

$\boxed{\text{Puntos } P_1 \text{ y } P_2}$

$$HL(x,y) = \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = (x_1, x_2) \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -6x_1^2$$

Puntos que anulan $\nabla g(x,y) = (8x, 2y)$

P_1 $\nabla g(P_1) = (0, -4)$

$$\nabla g(P_1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4x_2 = 0 \Rightarrow x_2 = 0$$

$C = -6x_1^2 < 0 \longrightarrow$ P_1 es un máximo relativo condicionado

↳ la desigualdad es estricta porque se evaluan en puntos $(x_1, x_2) \neq 0$ tales que $x_2 = 0$, por lo tanto $x_1 \neq 0$

P_2 $\quad \nabla g(P_2) = (0;4)$

$$\nabla g(P_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4x_2 = 0 \Rightarrow x_2 = 0$$

$C = -6x_1^2 < 0$

↳ la desigualdad es estricta porque se evaluan en puntos $(x_1, x_2) \neq 0$ tales que $x_2 = 0$, por lo tanto $x_1 \neq 0$

→ P_2 es un máximo relativo condicionado

$\boxed{\text{Puntos } P_3 \text{ y } P_4}$

$$HL(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 3/2 \end{pmatrix}$$

$$C = (x_1, x_2)\begin{pmatrix} 0 & 0 \\ 0 & 3/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{3}{2}x_2^2$$

Puntos que anulan $\nabla g(x,y) = (8x, 2y)$

P_3 $\quad \nabla g(P_3) = (8,0)$

$$\nabla g(P_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8x_1 = 0 \Rightarrow x_1 = 0$$

$C = \frac{3}{2}x_2^2 > 0$ → P_3 es un mínimo relativo condicionado

↳ la desigualdad es estricta porque se evaluan en puntos $(x_1, x_2) \neq 0$ tales que $x_1 = 0$, por lo tanto $x_2 \neq 0$

P_4 $\nabla g(P_4) = (-8, 0)$

$$\nabla g(P_1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -8x_1 = 0 \Rightarrow x_1 = 0$$

$C = \frac{3}{2}x_2^2 \geq 0 \longrightarrow P_4$ es un mínimo relativo condicionado

$\hookrightarrow$ la desigualdad es estricta porque se evalúan en puntos $(x_1, x_2) \neq 0$ tales que $x_1 = 0$, por lo tanto $x_2 \neq 0$

9.26

$f: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad$ $g: \mathbb{R}^3 \to \mathbb{R}^2$

$f(x,y) = (x + xe^y,\ x+y,\ y^2 \operatorname{sen} x)$ $\qquad$ $g(x,y,z) = (x + e^{xyz},\ y - xz)$

a) $$Jf(x,y) = \begin{pmatrix} 1+e^y & xe^y \\ 1 & 1 \\ y^2 \cos x & 2y \operatorname{sen} x \end{pmatrix}$$

b) $$Jg(x,y,z) = \begin{pmatrix} 1 + yz e^{xyz} & xz\, e^{xyz} & xy\, e^{xyz} \\ -z & 1 & -x \end{pmatrix}$$

c)

$$J(g \circ f)(0,0) = Jg(f(0,0)) \cdot Jf(0,0) = Jg(0,0,0)\, Jf(0,0)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

d) $$J(f \circ g)(0,0,0) = Jf(g(0,0,0)) \cdot Jg(0,0,0) = Jf(1,0)\, Jg(0,0,0) =$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

e) $$\frac{\partial g \circ f}{\partial x}(0,0) = (2,1)$$

$$\frac{\partial f \circ g}{\partial y}(0,0,0) = (1,1,0)$$

9.27 $h: \mathbb{R}^{2006} \longrightarrow \mathbb{R}^{2006}$

a) $J h^{-1}(h(a)) = (J h(a))^{-1}$

b) $H h(a) \in M_{2006 \times 2006}(\mathbb{R})$

$J h(a) \in M_{2006 \times 2006}(\mathbb{R})$ } coinciden los tamaños

c) Sí, es simétrica la matriz hessiana por ser h de clase C^2.

9.28

$f(x,y) = (x+y^2x, \; xy)$

$g(x,y) = (x+e^{xy}, \; y-x, \; \operatorname{sen}(xy))$

a) $Jf(x,y) = \begin{pmatrix} 1+y^2 & 2yx \\ y & x \end{pmatrix}$

b) $Jg(x,y) = \begin{pmatrix} 1+ye^{xy} & xe^{xy} \\ -1 & 1 \\ y\cos(xy) & x\cos(xy) \end{pmatrix}$

c) $g\circ f(x,y) = g(x+y^2x, \; xy) =$

$= \left(x+y^2x+e^{(x+y^2x)xy}, \; xy-x-y^2x, \; \operatorname{sen}[(x+y^2x)xy]\right)$

d) $Jg\circ f(0,1) = Jg(f(0,1)) \cdot Jf(0,1) = Jg(0,0)\, Jf(0,1) =$

$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}$

9.29 $h: \mathbb{R}^{2000} \longrightarrow \mathbb{R}^{2000}$

a) $Jh(a) \in M_{2000\times 2000}(\mathbb{R})$

b) No necesariamente tiene que ocurrir que $\frac{\partial h_i}{\partial x_j}(\vec{a}) = \frac{\partial h_j}{\partial x_i}(\vec{a})$

Por ejemplo, sea $h\left((x_i)_{i=1}^{2000}\right)$ definida por

$$h_j\left((x_i)_{i=1}^{2000}\right) = j\, x_{j+1} \quad ; \quad h_{2000}((x_i)) = 2000x_1$$

$$\frac{\partial f_i}{\partial x_{i+1}}(\vec{0}) = i \neq \frac{\partial f_{i+1}}{\partial x_i}(\vec{0}) = 0$$

$$h(x_1, x_2, x_3, x_4 \dots x_{2000}) = (x_2, 2x_3, 3x_4, 4x_5, \dots\dots, 2000x_1)$$

c) Que h sea de clase C^2.

9.30 $f: \mathbb{R}^n \to \mathbb{R}^m$

$$Jf(a) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = A$$

$|A| = 4-1 = 3 \neq 0$

a) $n = m = 2$

b) sí porque A es invertible y se puede aplicar el teorema de la función inversa

c) No necesariamente. Por ejemplo si tomamos

$h: \mathbb{R}^2 \to \mathbb{R}^2$ dado por $h(x,y) = (2\cos x + \operatorname{sen} y, \cos x + 2 \operatorname{sen} y)$

y $a = (0,0)$ entonces

$$Jh(0,0) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Sin embargo h no es globalmente invertible porque no es inyectiva ya que $h(x,y) = h(x+2\pi, y+2\pi)$.

d) $$Jf^{-1}(f(a)) = \left(Jf(a)\right)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{t} =$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

9.31 $f(x,y) = \operatorname{sen} x + \cos y \qquad (x,y) \in (0,\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\nabla f(x,y) = (\cos x, -\operatorname{sen} y) = (0,0)$$

$$\Rightarrow \begin{cases} \cos x = 0 \Rightarrow x = \frac{\pi}{2} \\ \operatorname{sen} y = 0 \Rightarrow y = 0 \end{cases}$$

$$Hf(0,0) = \begin{pmatrix} -\operatorname{sen}\left(\frac{\pi}{2}\right) & 0 \\ 0 & -\cos(0) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$1, \Delta_1, \Delta_2 \rightarrow 1, -1, 1$$

Así que $\left(\frac{\pi}{2}, 0\right)$ es un máximo relativo de f.

9.32 $f(x,y) = x^2 + 2x^2y^2 + 3y^2$

$$\nabla f(x,y) = (2x + 4xy^2,\ 4x^2y + 6y) = (0,0)$$

$$\Rightarrow \begin{cases} 2x + 4xy^2 = x(2+4y^2) = 0 \ \begin{cases} x = 0 \\ 2+4y^2 = 0 \end{cases} \\ 4x^2y + 6y = y(4x^2+6) = 0 \ \begin{cases} y = 0 \\ 4x^2+6 = 0 \end{cases} \end{cases}$$

Así que el único punto que anula el gradiente es el $(0,0)$.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 + 4y^2 \qquad \frac{\partial^2 f}{\partial y^2} = 4x^2 + 6$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 8xy$$

$$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

$$1,\ \Delta_1,\ \Delta_2 \rightsquigarrow 1,\ 2,\ 12$$

Así que en $(0,0)$ tenemos un mínimo relativo, que en este caso se puede ver fácilmente que es un mínimo absoluto.

9.37 $f(x,y) = x+y$

$A = \{(x,y)$ tq $x^2+y^2 \leq 4$, $x \leq 0\}$

La existencia de extremos absolutos está garantizada por estar f definida en el compacto A y ser continua.

a. Buscamos los extremos relativos en el interior de A

$\nabla f(x,y) = (1,1) \neq (0,0)$, luego no hay extremos relativos.

b. Buscamos extremos condicionados por $x^2+y^2=4$

$$L(x,y) = x+y+\lambda(x^2+y^2-4)$$

$$\nabla L(x,y) = (1+2\lambda x,\ 1+2\lambda y) = (0,0) \Rightarrow \left\{\begin{array}{l} 1+2\lambda x = 0 \\ 1+2\lambda y = 0 \end{array}\right\} = 2\lambda x = 2\lambda y$$

$$\Rightarrow \left\{\begin{array}{ll} \lambda = 0 & \text{(imposible)} \\ \text{ó} & \\ x = y & \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2} = y \end{array}\right.$$

$$P_1 = (\sqrt{2}, \sqrt{2}) \rightarrow \lambda = \frac{-1}{2x}$$

$$P_2 = (-\sqrt{2}, -\sqrt{2}) \nearrow$$

$$\frac{\partial^2 L}{\partial x^2} = 2\lambda \qquad \frac{\partial^2 L}{\partial y^2} = 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y} = \frac{\partial^2 L}{\partial y \partial x} = 0$$

$$HL(x,y) = \begin{pmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{pmatrix}$$

$$(h_1, h_2)\, HL(x,y)\, (h_1, h_2)^t = 2\lambda(h_1^2 + h_2^2) = Q$$

P_1 $\quad Q < 0 \Rightarrow P_1$ es un máximo relativo condicionado

P_2 $\quad Q > 0 \Rightarrow P_2$ " " mínimo relativo "

C. Buscamos extremos condicionados por $x = 0$

$f(x,y) = f(0,y) = y$ (no tiene extremos)

D Comparamos todos los puntos:

$f(P_1) = 2\sqrt{2}$

$f(P_2) = -2\sqrt{2}$

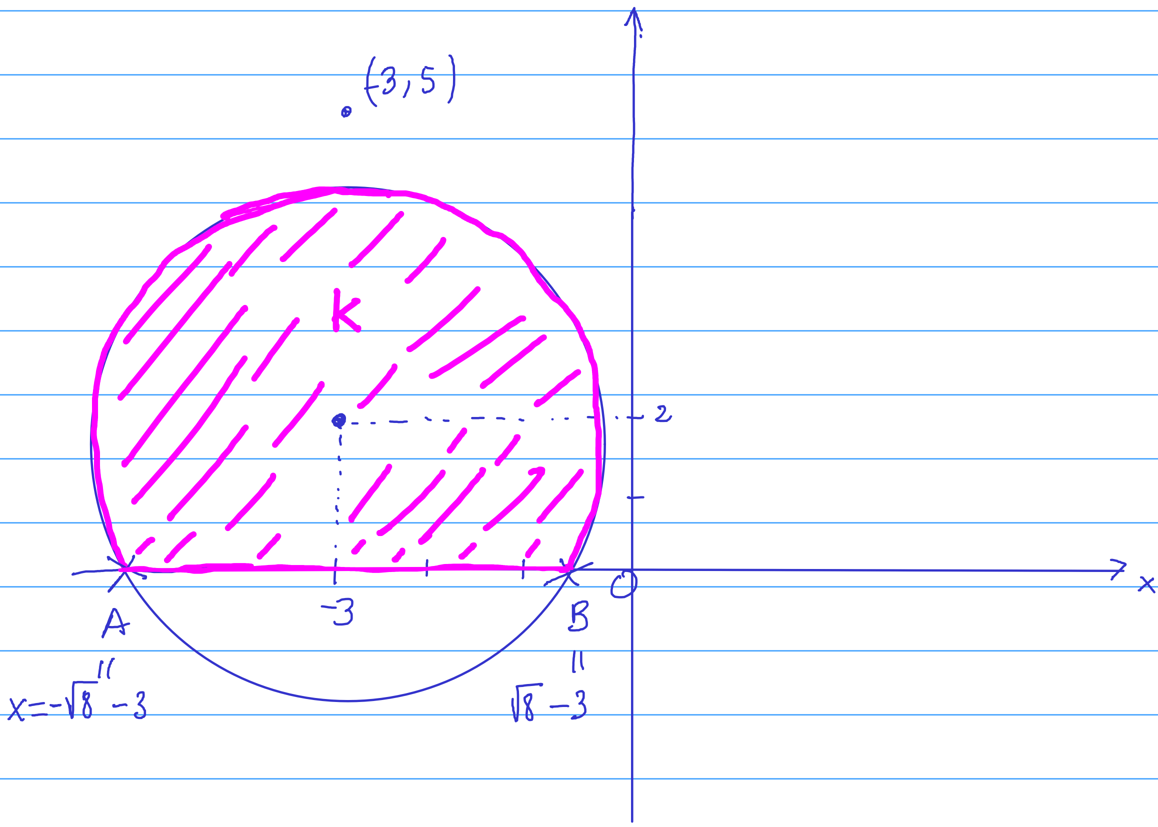
$f(0,2) = 2$

$f(0,-2) = -2$

→ estos puntos se meten por cortar la ligadura.

Así que P_1 es el máximo absoluto y P_2 es el mínimo absoluto.

9.43 $K = \{ (x,y) : (x+3)^2 + (y-2)^2 \leq 8 , y \geq 0 \}$



Sea (x,y) un punto del conjunto K.

$f(x,y) = d((x,y),(-3,5)) = \sqrt{(x+3)^2 + (y-5)^2}$

Los máximo de $f(x,y)$ se situarán en los mismos puntos que los de $g(x,y) = (f(x,y))^2 = (x+3)^2 + (y-5)^2$

Buscamos los extremos relativos en Int K

$$\nabla g(x,y) = \big(2(x+3) , 2(y-5) \big) = (0,0) \Rightarrow \begin{cases} x = -3 \\ y = 5 \end{cases}$$

Pero $(-3,5) \notin \text{Int } K$, con lo cual no tenemos puntos candidatos a extremos en Int K.

Buscamos ahora en los puntos de la frontera.

$\partial_1 K$ $\quad \partial_1 K = \{(x,y) \text{ tq } (x+3)^2 + (y-2)^2 = 8, \, y > 0\}$

$$L(x,y) = (x+3)^2 + (y-5)^2 + \lambda\left[(x+3)^2 + (y-2)^2 - 8\right]$$

$$\frac{\partial L}{\partial x} = (\lambda+1)\,2\,(x+3) \begin{cases} x = -3 \\ \lambda = -1 \end{cases}$$

$$\frac{\partial L}{\partial y} = 2(y-5) + 2\lambda(y-2) = 0 = (2+2\lambda)y - 10 - 4\lambda = 0$$

$$(x+3)^2 + (y-2)^2 - 8 = 0$$

(*) $x = -3 \Rightarrow (y-2)^2 = 8 \Rightarrow y - 2 = \pm\sqrt{8} \Rightarrow \begin{cases} y = 2+\sqrt{8} \\ \cancel{y = +2-\sqrt{8}} \end{cases}$

$$(2+2\lambda)(2+\sqrt{8}) - 4\lambda = 10$$

$$\Rightarrow \quad 2\sqrt{8}\,\lambda = 6 - 2\sqrt{8} \Rightarrow \lambda = \frac{3}{\sqrt{8}} - 1$$

(*) $\lambda = -1 \Rightarrow -10 = 4\lambda \Rightarrow \lambda = \frac{-10}{4}$!! (no hay solución para $\lambda = -1$)

$$HL(x,y) = \begin{pmatrix} 2(\lambda+1) & 0 \\ 0 & 2(\lambda+1) \end{pmatrix}$$

$(x,y) = (-3,\ 2+\sqrt{8})$ $\qquad (h_1,h_2) \neq (0,0)$

$$(h_1,h_2)\ HL(x,y) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 2(\lambda+1)(h_1^2 + h_2^2) > 0 \Rightarrow (**)$$

(h_1,h_2) verifica $\qquad \nabla g(x,y) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 2\sqrt{8}\, h_2 = 0 \Rightarrow h_2 = 0$

$$\nabla g(x,y) = \left(2(x+3) \quad 2(y-2)\right)$$

$(**)$ $\quad (-3,\ 2+\sqrt{8})$ es un mínimo relativo condicionado

$\partial_2 K$ $\qquad \partial_2 K = \{(x,y) : y=0,\ A<y<B\}$

$$g(x,y) \overset{y=0}{=} (x+3)^2 = G(x)$$

$$G'(x) = 2(x+3) = 0 \Rightarrow x=-3 \qquad y=0$$

$$G''(x) = 2 > 0$$ $\Rightarrow$ $(-3,0)$ es un mínimo relativo condicionado.

$\partial_3 K$ $\qquad \partial_3 K = \{(A,0)\}$

$\partial_4 K$ $\qquad \partial_4 K = \{(B,0)\}$

puntos candidatos a extremos

Estudiamos los 4 puntos candidatos a extremos absolutos

$$g(\sqrt{8}-3,0) = 8+25 = 33$$

$$g(-\sqrt{8}-3,0) = 8+25 = 33$$

$$g(-3,0) = 0+25 = 25$$

$$g(-3,\ 2+\sqrt{8}) = 0 + (-3+\sqrt{8})^2 = 9+8-6\sqrt{8} = 17-6\sqrt{8}$$

Así que:

$(8-\sqrt{3},0)$ y $(-8-\sqrt{3},0)$ son máximos absolutos

$(-3,2+\sqrt{8})$ es un mínimo absoluto.

9.45

$f(x,y,z) = x^2+y^2+z^2$

ligadurs $\begin{cases} x^2+y^2=1 & \longrightarrow \quad g_1(x,y,z) = x^2+y^2-1 \\ x+y+z=1 & \longrightarrow \quad g_2(x,y,z) = x+y+z-1 \end{cases}$

$L(x,y,z) = x^2+y^2+z^2+\lambda(x^2+y^2-1)+\mu(x+y+z-1)$

$$(S)\begin{cases} \frac{\partial L}{\partial x} = 2x+2\lambda x+\mu = 2x(1+\lambda)+\mu=0 \\ \frac{\partial L}{\partial y} = 2y+2\lambda y+\mu = 2y(1+\lambda)+\mu=0 \\ \frac{\partial L}{\partial z} = 2z+\mu=0 \\ x^2+y^2=1 \\ x+y+z=1 \end{cases}$$

si $\lambda\neq-1$

$2x(1+\lambda) = 2y(1+\lambda) \Rightarrow x=y \Rightarrow 2x^2=1 \Rightarrow y=x=\pm\frac{1}{\sqrt{2}} \Rightarrow$

$\Rightarrow z = 1\mp\frac{2}{\sqrt{2}} \Rightarrow \mu=-2z = -2\pm\frac{4}{\sqrt{2}} \Rightarrow$

$\Rightarrow \lambda = \frac{-\mu}{2y}-1 = \frac{2\mp\frac{4}{\sqrt{2}}}{2\cdot\frac{1}{\pm\sqrt{2}}}-1 = \pm\sqrt{2}-2-1 = \pm\sqrt{2}-3$

si $\lambda=-1 \Rightarrow \mu=0 \Rightarrow z=0 \Rightarrow \left.\begin{cases} x+y=1 \\ x^2+y^2=1 \end{cases}\right\} \Rightarrow x^2+(1-x)^2=1$

$\Rightarrow 2x^2-2x+1=1 \Rightarrow 2x(x-1)=0 \Rightarrow \begin{cases} x=0 \Rightarrow y=1 \\ x=1 \Rightarrow y=0 \end{cases}$

SOLUCIONES DE (S)

A. $x=0,\ y=1,\ z=0$ con $\lambda=-1$ y $\mu=0$

B. $x=1,\ y=0,\ z=0$ con $\lambda=1$ y $\mu=0$

C. $x=y=\dfrac{1}{\sqrt{2}},\ z=1-\sqrt{2}$ con $\lambda=\sqrt{2}-3$ y $\mu=-2+2\sqrt{2}$

D. $x=y=\dfrac{-1}{\sqrt{2}},\ z=1+\sqrt{2}$ con $\lambda=-\sqrt{2}-3$ y $\mu=-2-2\sqrt{2}$

Calculamos ahora $HL(x,y,z)$

$$\frac{\partial L}{\partial x}=2x+2\lambda x+\mu=2x(1+\lambda)+\mu$$

$$\frac{\partial L}{\partial y}=2y+2\lambda y+\mu=2y(1+\lambda)+\mu$$

$$\frac{\partial L}{\partial z}=2z+\mu=0$$

$$\frac{\partial^2 L}{\partial x^2}=2(1+\lambda)$$

$$\frac{\partial^2 L}{\partial y^2}=2(1+\lambda)$$

$$\frac{\partial^2 L}{\partial z^2}=2$$

$$\frac{\partial^2 L}{\partial x\partial y}=\frac{\partial^2 L}{\partial x\partial z}=\frac{\partial^2 L}{\partial y\partial z}=0$$

$$HL(x,y,z)=\begin{pmatrix} 2(1+\lambda) & 0 & 0\\ 0 & 2(1+\lambda) & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$(h_1,h_2,h_3)\ HL(x,y,z)\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}=2(1+\lambda)(h_1^2+h_2^2)+2h_3^2=Q$$

(h_1,h_2,h_3) debe satisfacer:

C1. $\nabla g_1(x,y,z)\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}=(2x,2y,0)\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}=2x\,h_1+2y\,h_2=0$

c.2 $\nabla g_2(x,y,z)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = (1,1,1)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = h_1+h_2+h_3=0$

PUNTO A $\quad \lambda=-1 \quad A=(0,1,0)$

$Q = 2h_3^2 \Rightarrow Q>0$

$\left.\begin{array}{l} h_1+h_2+h_3=0 \\ 2h_2=0 \end{array}\right\} \Rightarrow h_1+h_3=0 \Rightarrow \left.\begin{array}{l} h_3=-h_1 \\ (h_1,h_2,h_3)\neq 0 \\ h_2=0 \end{array}\right\} \Rightarrow h_3\neq 0$

A es un mínimo relativo condicionado

PUNTO B $\quad \lambda=-1 \quad B=(1,0,0)$

$Q = 2h_3^2 \Rightarrow Q>0$

$\left.\begin{array}{l} h_1+h_2+h_3=0 \\ 2h_1=0 \end{array}\right\} \Rightarrow h_2+h_3=0 \Rightarrow \left.\begin{array}{l} h_3=-h_2 \\ (h_1,h_2,h_3)\neq 0 \\ h_1=0 \end{array}\right\} \Rightarrow h_3\neq 0$

B es un mínimo relativo condicionado

$\boxed{\text{PUNTO C}}$ $C=\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right)$ $\lambda=\sqrt{2}-3$ y $\mu=-2+2\sqrt{2}$

$$2(1+\lambda)\left(h_1^2+h_2^2\right)+2h_3^2 = Q$$

$$\| $$

$$2(\sqrt{2}-2)\left(h_1^2+h_2^2\right)+2h_3^2$$

$$2x\,h_1+2y\,h_2=0 = \frac{2}{\sqrt{2}}(h_1+h_2)=0 \Rightarrow \underline{h_1=-h_2}$$

$$h_1+h_2+h_3=0 \Rightarrow h_3=0$$

$$(h_1, h_2, h_3) \neq (0,0,0)$$

$$\left.\right\} \Rightarrow h_1 \neq 0 \neq h_2$$

$$Q = 2\underbrace{(\sqrt{2}-2)}_{\substack{\wedge\\0}}\underbrace{\left(h_1^2+h_2^2\right)}_{\substack{\vee\\0}} < 0 \Rightarrow$$

C es un máximo relativo condicionado

$\boxed{\text{PUNTO D}}$ $D = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$ $\lambda = -\sqrt{2}-3$ y $\mu = -2+2\sqrt{2}$

$$2(1+\lambda)(h_1^2+h_2^2) + 2h_3^2 = Q$$

$$\parallel$$

$$2(\sqrt{2}-4)(h_1^2+h_2^2) + 2h_3^2$$

$$\left.\begin{array}{l} 2x\,h_1 + 2y\,h_2 = 0 = \frac{-2}{\sqrt{2}}(h_1+h_2) = 0 \Rightarrow \underline{h_1 = -h_2} \\ h_1+h_2+h_3 = 0 \Rightarrow h_3 = 0 \\ \qquad (h_1, h_2, h_3) \neq (0,0,0) \end{array}\right\} \Rightarrow h_1 \neq 0 \neq h_2$$

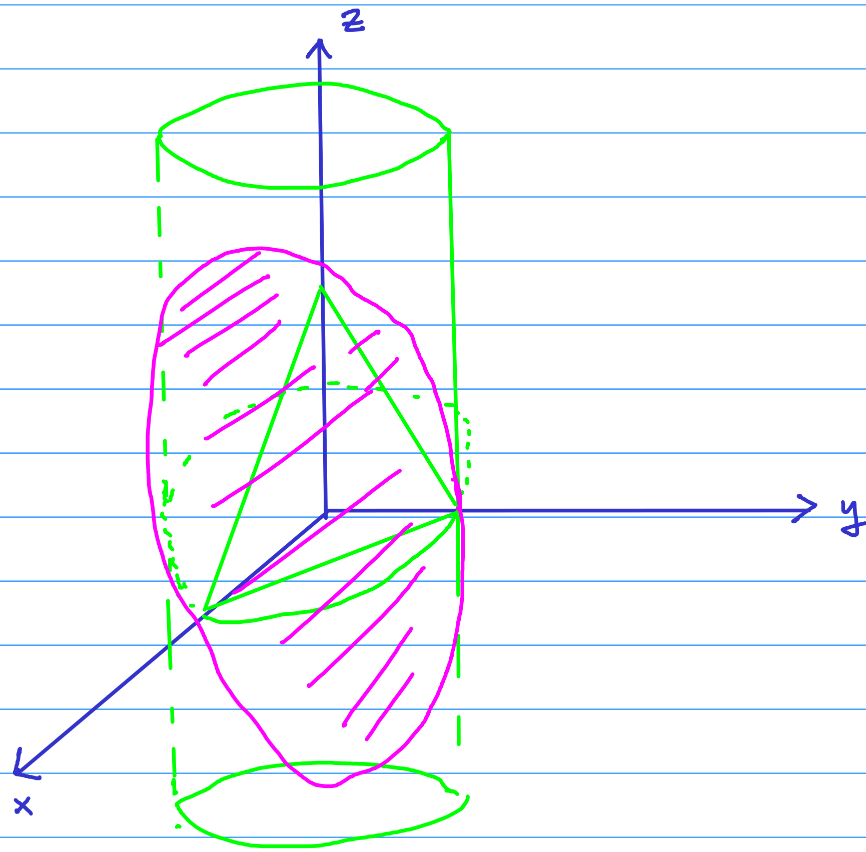
$$Q = 2\underbrace{(\sqrt{2}-4)}_{<0}\underbrace{(h_1^2+h_2^2)}_{>0} < 0 \Rightarrow$$ D es un máximo relativo condicionado

FINALMENTE, comparando lo que vale f en todos los puntos extremos relativos, tenemos:

A y B son mínimos absolutos
D es un máximo absoluto.

9.46

Este ejercicio es una continuación del anterior porque intersecamos el plano $x+y+z=1$ con el cilindro "macizo" $x^2+y^2\leq 1$. La intersección será la elipse del ejercicio anterior junto con la porción de plano $x+y+z=1$ delimitada por ella.



Así que habrá que buscar los extremos relativos condicionados dentro de la elipse y en el borde. La segunda cosa ya está estudiada en el ejercicio anterior, abordamos la primera y posteriormente veremos qué puntos son los extremos absolutos.

Estudiamos los extremos relativos de $f(x,y,z) = x^2+y^2+z^2$ condicionada por $x+y+z-1 = g_2(x,y,z)$

$$L(x,y,z) = x^2+y^2+z^2+\lambda(x+y+z-1)$$

$$\left.\begin{array}{l} \frac{\partial L}{\partial x} = 2x+\lambda \\ \frac{\partial L}{\partial y} = 2y+\lambda \\ \frac{\partial L}{\partial z} = 2z+\lambda \\ x+y+z=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x=y=z \\ x+y+z=1 \end{array}\right\} \Rightarrow \begin{array}{l} x=y=z=\frac{1}{3} \\ \lambda = -2/3 \end{array}$$

El único punto candidato es $E = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Estudiamos la matriz hessiana:

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial^2 L}{\partial y^2} = \frac{\partial^2 L}{\partial z^2} = 2 \qquad \frac{\partial^2 L}{\partial x \partial y} = \frac{\partial^2 L}{\partial x \partial z} = \frac{\partial^2 L}{\partial y \partial z} = 0$$

$$HL(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$Q = (h_1, h_2, h_3)\, HL(x,y,z) \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 2(h_1^2+h_2^2+h_3^2) > 0 \quad (h_1,h_2,h_3) \neq 0$$

Así que E es un mínimo relativo condicionado.

Como $d(\underset{\underset{(0,0,0)}{\parallel}}{E, O}) = \sqrt{\frac{3}{9}} = \frac{1}{\sqrt{3}} < 1 = d(A, O)$, obtenemos:

E es el mínimo absoluto y D el máximo absoluto.

9.46 Elipse : $5x^2+6xy+5y^2=8$

Buscamos extremos de $f(x,y) = x^2+y^2$ sujeto a la ligadura $g(x,y) = 5x^2+6xy+5y^2-8$

$L(x,y) = x^2+y^2+\lambda(5x^2+6xy+5y^2-8)$

$\frac{\partial L}{\partial x} = 2x + 10\lambda x + 6y$

9.48

$f(x,y,z) = xyz$

$\nabla f(x,y,z) = (yz, xz, xy) = \vec{0} \Rightarrow$

$$\Rightarrow \begin{cases} yz = 0 & y=0 \text{ ó } z=0 \\ xz = 0 & x=0 \text{ ó } z=0 \\ xy = 0 & x=0 \text{ ó } y=0 \end{cases} \Rightarrow \begin{cases} \bullet\ y=0,\ x=0 \\ \bullet\ y=0,\ z=0 \\ \bullet\ z=0,\ x=0 \end{cases}$$

Así que los puntos candidatos a extremos son:

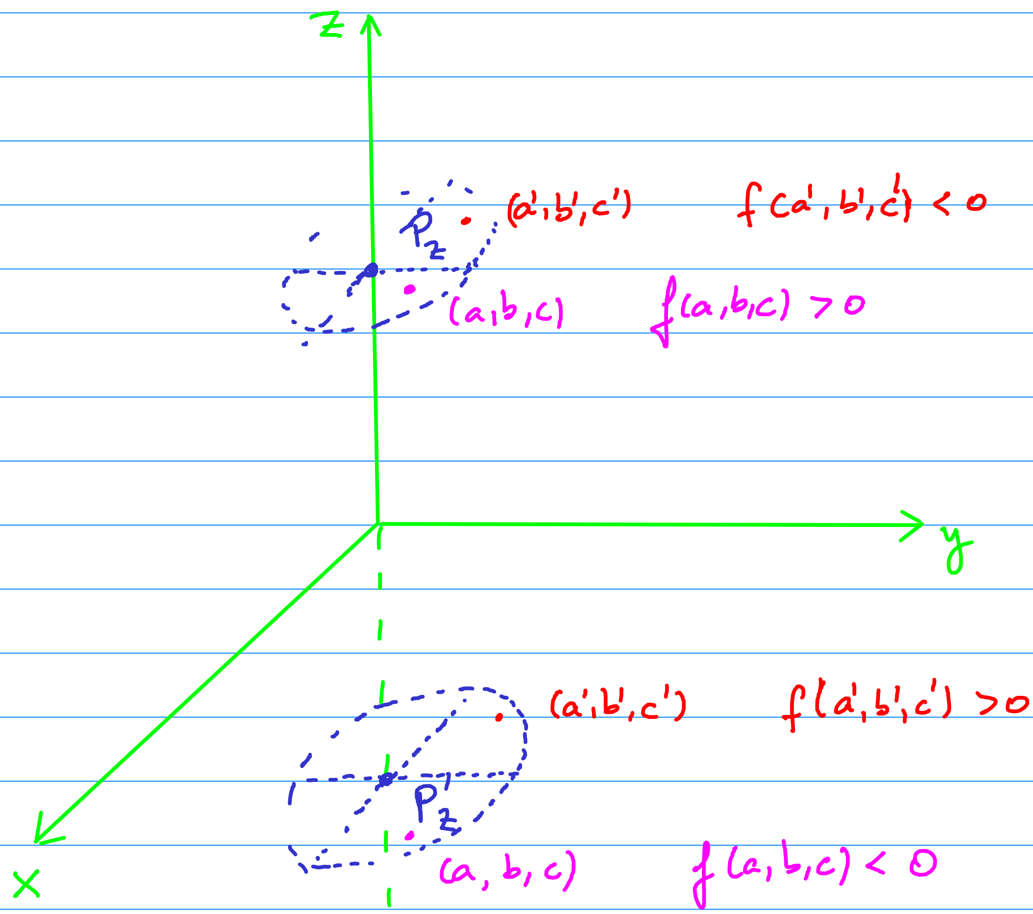
$P_x = (x,0,0)$; $P_y = (0,y,0)$ y $P_z = (0,0,z)$ $\forall (x,y,z) \in \mathbb{R}^3$.

Si el estudiante estudia el criterio del hessiano verá que no se le puede aplicar.

Sin embargo cualitativamente podemos ver que para $P_z = (0,0,z)$ $f(P_z) = 0$. Además podemos encontrar cerca de él puntos con todas sus coordenadas distintas de 0 con ternas (a,b,c) cercanas a P_z y tales que unas tienen signo$(a) > 0$, signo$(b) > 0$, signo$(c) =$ signo(z)

y otros en los que signo$(a) < 0$, signo$(b) > 0$ y signo$(c) =$ signo(z)

Luego P_z no puede ser un extremo relativo. El razonamiento se puede extender a los puntos P_x y P_y de manera sencilla. Este razonamiento puede quedar más claro analizando el dibujo siguiente:



9.49

$$\left.\begin{array}{l} f(x,y,z) = xyz \\ x+y+z=27 \end{array}\right\}$$

$$L(x,y,z) = xyz + \lambda(x+y+z-27)$$

$$\nabla L(x,y,z) = (yz+\lambda, xz+\lambda, xy+\lambda) = (0,0,0) \Rightarrow$$

$$\Rightarrow (S) \left\{ \begin{array}{l} yz+\lambda=0 \\ xz+\lambda=0 \\ xy+\lambda=0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} yz = xz \begin{cases} z=0 \\ \text{ó} \\ z\neq 0 \quad x=y \end{cases} \\ yz = xy \begin{cases} y=0 \\ y\neq 0 \quad z=x \end{cases} \end{array} \right.$$

Así que las soluciones de (S) añadiendo $x+y+z=27$ son:

* $z=0$, $y=0$, $x=27$	$P_1 = (27,0,0)$	$\lambda=0$
* $z=0$, $y\neq 0$, $z=x=0$, $y=27$	$P_2 = (0,27,0)$	$\lambda=0$
* $z\neq 0$, $x=y=0$	$P_3 = (0,0,27)$	$\lambda=0$
* $z\neq 0$, $x=y=z$, $y\neq 0$	$P_4 = (9,9,9)$	$\lambda=-81$

Calculamos la matriz Hessiana

$$HL(x,y,z) = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

Estudiamos el punto P_4. La función condición es $g(x,y,z) = x+y+z-27$, luego $\nabla g(x,y,z) = (1,1,1)$. Analizamos qué vectores cumplen $\nabla g(x,y,z)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 0$ y éstos verifican $h_1+h_2+h_3=0$, es decir, $h_3 = -h_1-h_2$.

Ahora analizamos el signo de la cantidad

$$Q = (h_1\ h_2\ h_3)\ HL(P_4) \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 18(h_1h_2 + h_2h_3 + h_1h_3)$$

para los vectores $(h_1, h_2\ h_3) \neq \vec{0}$ con $h_3 = -h_1 - h_2$.

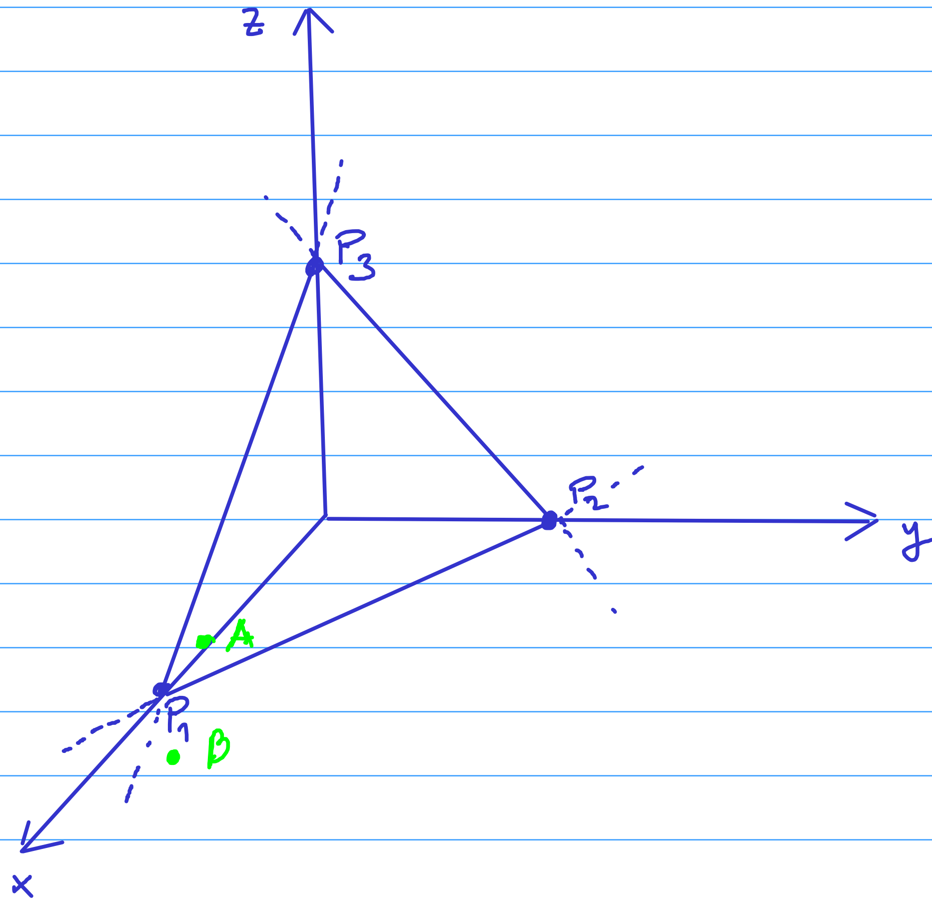
$$Q = 18[h_1h_2 + h_2(-h_1-h_2) + h_1(-h_1-h_2)] = -18[h_1^2 + h_2^2 - h_1h_2]$$

Observemos que h_1 y h_2 no pueden ser simultáneamente nulos porque $(h_1, h_2, h_3) \neq \vec{0}$ y $h_3 = -h_1 - h_2$. Además:

$$Q = -18\underbrace{[h_1^2 + h_2^2 - h_1h_2]}_{Q'} = -18\underbrace{[(h_1-h_2)^2 + h_1h_2]}_{Q''}$$

Observa que si h_1 y h_2 tienen el mismo signo $Q'' > 0$ y si h_1 y h_2 tienen signo diferente $Q' > 0$, luego $Q < 0$ y P_4 es un máximo relativo condicionado.

$\boxed{\text{Estudiamos el punto } P_1 = (27, 0, 0)}$ El camino del hessiano no conduce a ningún lado porque la cantidad Q se puede anular en este caso. Hay que analizar lo que pasa en este caso en las proximidades de P_1



Los puntos $A \doteq (a_1, a_2, a_3)$ y $B = (b_1, b_2, b_3)$ los cogemos en el plano $x+y+z=27$ con $\underbrace{a_1, a_2, a_3}_{>0}, \underbrace{b_1, b_2}, \underbrace{b_3}_{<0} \neq 0$

Como $f(A) > 0$, $f(P_1) = 0$ y $f(B) < 0$ se tiene que P_1 no puede ser un extremo relativo condicionado.