Integración de funcions de varias variables

10.1 $\Omega = [0,1] \times [0,3]$

a) $\iint_\Omega xy\,dx\,dy = \int_0^1 \left(\int_0^3 xy\,dy\right)dx = \int_0^1 x\left(\int_0^3 y\,dy\right)dx$

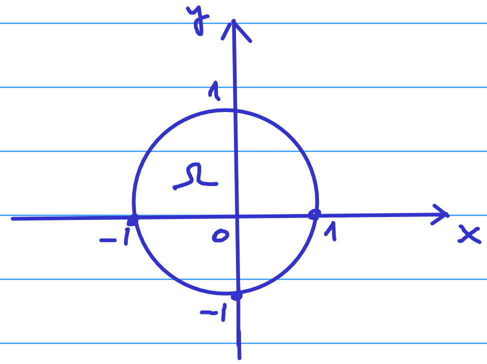
$= \int_0^1 x \left[\frac{y^2}{2}\right]_{y=0}^{y=3} dx = \frac{9}{2}\int_0^1 x\,dx = \frac{9}{2}\cdot\frac{1}{2} = \frac{9}{4}$

b) $\iint_\Omega x e^y\,dx\,dy = \int_0^3 \left(\int_0^1 x e^y\,dx\right)dy = \int_0^3 e^y\left(\int_0^1 x\,dx\right)dy$

$= \frac{1}{2}\int_0^3 e^y\,dy = \frac{1}{2}(e^3 - e^0) = \frac{1}{2}(e^3-1).$

10.2

a) $\Omega = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1 \}$



$$\iint_\Omega y \, dx\, dy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y \, dy \right) dx = \int_{-1}^{1} \left[\frac{y^2}{2} \right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left(\frac{1-x^2}{2} - \frac{1-x^2}{2} \right) dx = \int_{-1}^{1} 0 \, dx = 0$$

b) $\Omega = \{ (x,y) \text{ tq } x^2+y^2 \leq 1 \}$

$$\iint_\Omega (3y^3+x) \, dx\, dy = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3y^3+x \; dy \right) dx =$$

$$= \int_{-1}^{1} \left[3\frac{y^4}{4} + xy \right]_{y=-\sqrt{1-x^4}}^{y=\sqrt{1-x^2}} dx = -\int_{-1}^{1} -2x\sqrt{1-x^2} \, dx =$$

$$= - \left[\frac{(1-x^2)^{3/2}}{3/2} \right]_{x=-1}^{x=1} = 0$$

c) $\iint_\Omega \sqrt{xy}\,dx\,dy \qquad \Omega = \{ (x,y) \in \mathbb{R}^2 \text{ tq } \ 0 \le y \le 1 ,\ y^2 \le x \le y \}$

$$\| $$

$$\int_0^1 \left(\int_{y^2}^{y} \sqrt{xy}\, dx \right) dy = \int_0^1 \sqrt{y} \left[\frac{x^{3/2}}{3/2} \right]_{x=y^2}^{x=y} dy =$$

$$= \frac{2}{3} \int_0^1 \sqrt{y} \left(y^{3/2} - y^3 \right) dy = \frac{2}{3} \int_0^1 \left(y^2 - y^{7/2} \right) dy = \frac{2}{3} \left[\frac{y^3}{3} - \frac{y^{9/2}}{9/2} \right]_{y=0}^{y=1}$$

$$= \frac{2}{3} \left(\frac{1}{3} - \frac{2}{9} \right) = \frac{2}{3} \ \frac{1}{9} = \frac{2}{27}$$

d) $\Omega = \{ (x,y) : \ 0 \le y \le 1 , \ 0 \le x \le y^2 \}$

$$\iint_\Omega y e^x \, dx\, dy = \int_0^1 \left(\int_0^{y^2} y e^x \, dx \right) dy = \int_0^1 y \left[e^x \right]_{x=0}^{x=y^2} dy$$

$$= \int_0^1 y \left(e^{y^2} - e^0 \right) dy = \frac{1}{2} \int_0^1 2y\, e^{y^2} dy - \int_0^1 y\, dy = \frac{1}{2} \left[e^{y^2} \right]_{y=0}^{y=1} - \left[\frac{y^2}{2} \right]_{y=0}^{y=1}$$

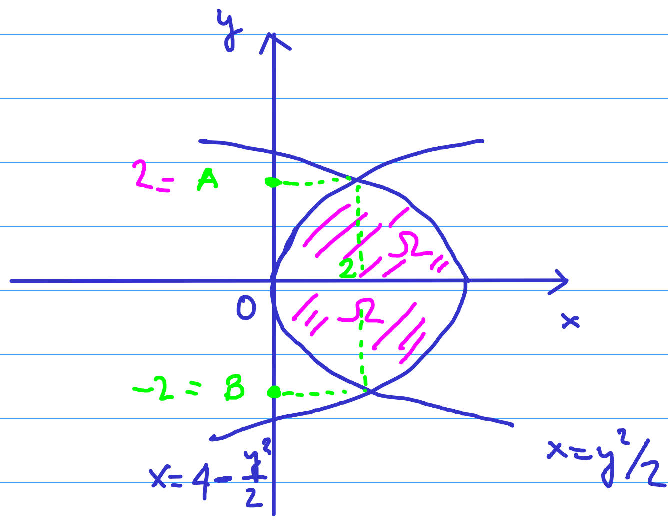
$$= \frac{1}{2} \left(e^1 - 1 \right) - \frac{1}{2} = \frac{e^1}{2} - 1$$

(10.3) a) $\iint_\Omega (4-y^2)\,dx\,dy$

Ω: recinto limitado por $y^2=2x$ e $y^2=8-2x$

$\Updownarrow$

$x=y^2/2 \qquad x=4-\dfrac{y^2}{2}$



Calculemos la intersección de las dos parábolas

$$\frac{y^2}{2}=4-\frac{y^2}{2} \iff y^2=4 \iff y=\pm 2$$

$$\iint_\Omega 4-y^2\,dx\,dy = \int_{-2}^{2}\left(\int_{y^2/2}^{4-\frac{y^2}{2}}(4-y^2)dx\right)dy =$$

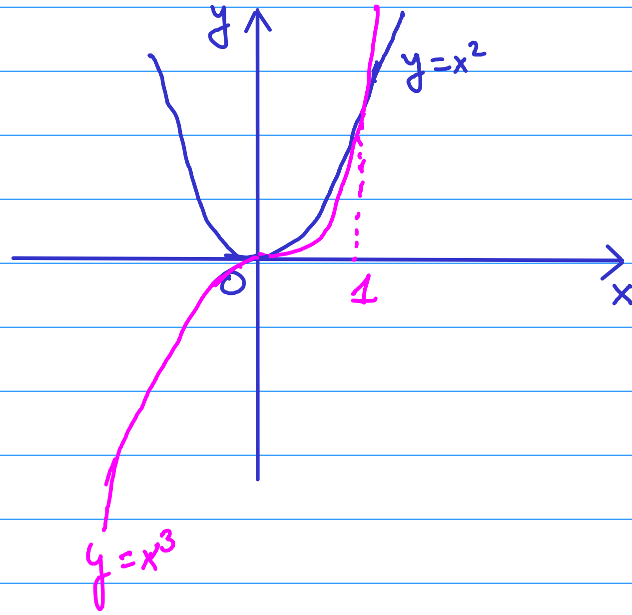
$$=\int_{-2}^{2}(4-y^2)\left(4-\frac{y^2}{2}-\frac{y^2}{2}\right)dy = \int_{-2}^{2}(4-y^2)^2\,dy =$$

$$=\int_{-2}^{2}(16+y^4-8y^2)\,dy = \left[16y-\frac{y^5}{5}-8\frac{y^3}{3}\right]_{y=-2}^{y=2} =$$

$$=64-\frac{2^6}{5}-\frac{8}{3}2^4 = 64-\frac{64}{5}-\frac{128}{3} = \frac{2\cdot 64}{15} = \frac{128}{15}$$

10.3.b.

Ω: recinto limitado por $y=x^3$ e $y=x^2$

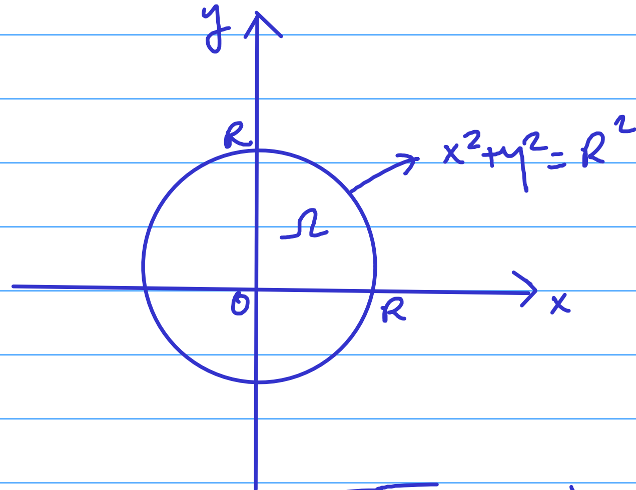


$$\iint_\Omega (x^4+y^2)\,dx\,dy = \int_0^1\left(\int_{x^3}^{x^2}(x^4+y^2)\,dy\right)dx = \int_0^1 \left[x^4 y + \frac{y^3}{3}\right]_{y=x^3}^{y=x^2} dx$$

$$= \int_0^1 \left[x^4\left(x^2-x^3\right)+\frac{1}{3}\left(x^6-x^9\right)\right]dx = \left[\frac{x^7}{7}-\frac{x^8}{8}+\frac{1}{3}\,\frac{x^7}{7}-\frac{1}{3}\,\frac{x^{10}}{10}\right]_{x=0}^{x=1}$$

$$= \frac{1}{7}-\frac{1}{8}+\frac{1}{21}-\frac{1}{30}$$

10.4.a) Círculo de radio R



$$A=\iint_\Omega 1\,dx\,dy = \int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} 1\,dy\right)dx =$$

se le pone − pq es $\sqrt{\ }$ positiva.

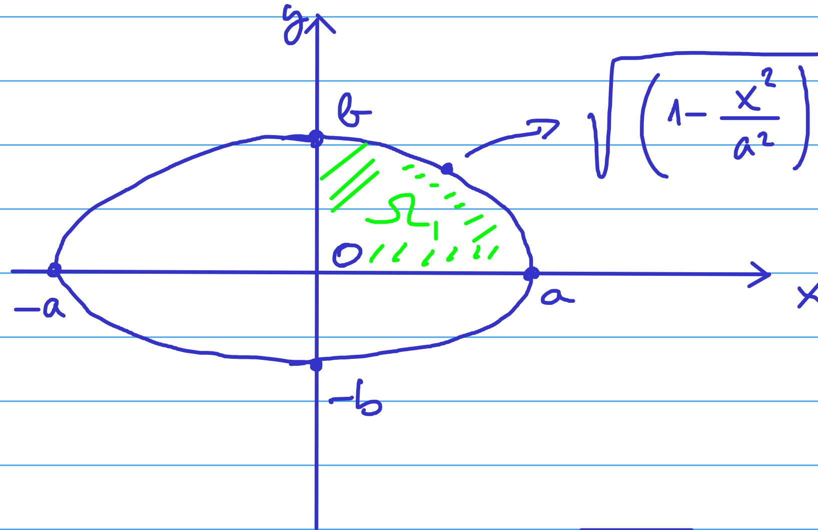
$$= \int_{-R}^{R} 2\sqrt{R^2-x^2}\,dx = \int_{-\pi}^{0} 2\,\overbrace{(-R\,\text{sen}\,\theta)}\,(-R)\,\text{sen}\,\theta\,d\theta \stackrel{*}{=}$$

$$x=R\cos\theta \Rightarrow dx=-R\,\text{sen}\,\theta\,d\theta$$

$$\stackrel{*}{=} 2R^2\int_{-\pi}^{0}\text{sen}^2\theta\,d\theta = 2R^2\int_{-\pi}^{0}\frac{1-\cos 2\theta}{2}\,d\theta = 2R^2\left[\frac{1}{2}\theta-\frac{\text{sen}\,2\theta}{4}\right]_{\theta=-\pi}^{\theta=0}$$

$$= 2R^2\left(\frac{1}{2}\pi-\frac{\text{sen}\,0}{4}+\frac{\text{sen}\,{-2\pi}}{4}\right) = \pi R^2$$

10.4.b) Elipse de semiejes a y b $\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



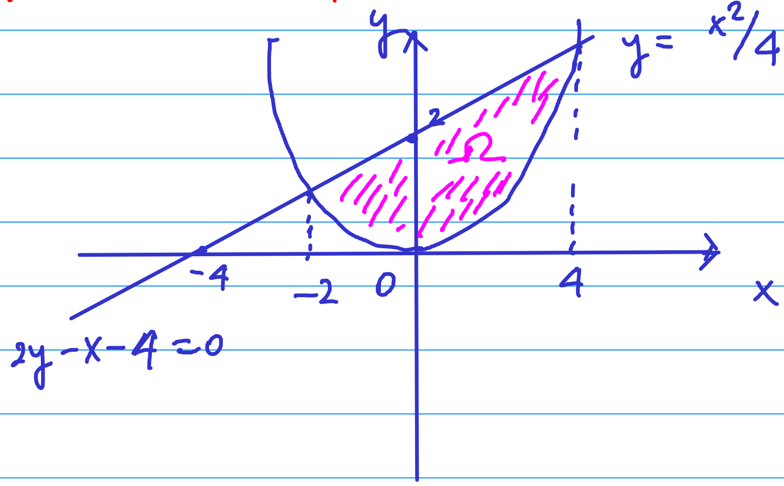
$$A = 4\iint_{\Omega_1} 1\,dx\,dy = 4\int_0^a\left(\int_0^{\sqrt{\left(1-\frac{x^2}{a^2}\right)b^2}} 1\,dy\right)dx =$$

$$= 4\int_0^a\left(b\sqrt{1-\frac{x^2}{a^2}}\right)dx = 4b\int_0^a\sqrt{1-\frac{x^2}{a^2}}\,dy \overset{\substack{x=a\,\mathrm{sen}\,\theta \\ dx=a\cos\theta\,d\theta}}{=}$$

$$= 4b\int_0^{\pi/2} a\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta =$$

$$= 4ab\left[\frac{\theta}{2}+\frac{\mathrm{sen}\,2\theta}{4}\right]_{\theta=0}^{\theta=\pi/2} = 4ab\left(\frac{\pi}{4}+0-0-0\right) = \pi ab.$$

10.4 b Ω: región limitada por $x^2 = 4y$ y $2y - x - 4 = 0$



$$\left.\begin{array}{l} x^2 = 4y \\ 2y - x - 4 = 0 \end{array}\right\} \quad \frac{x^2}{2} - x - 4 = 0 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow$$

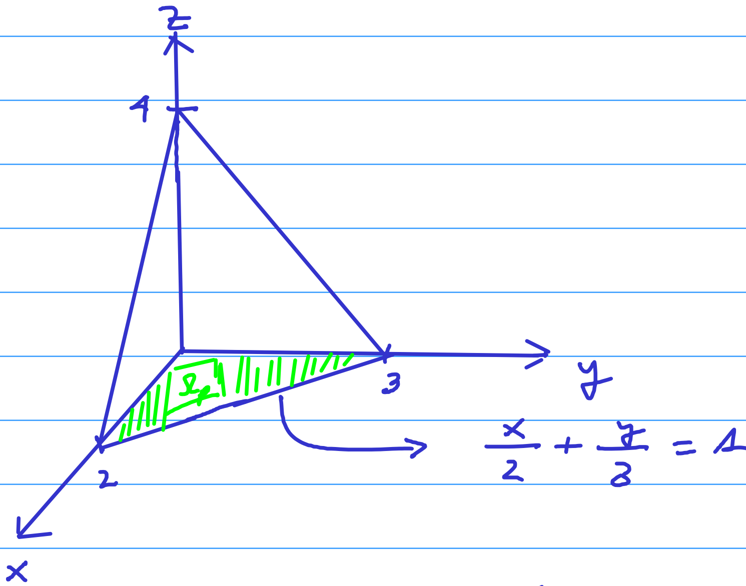
$$\Rightarrow x = \frac{2 \pm \sqrt{4 + 32}}{2} = \frac{2 \pm 6}{2} = \begin{cases} 4 \\ -2 \end{cases}$$

$$A = \iint_{\Omega} 1 \, dx dy = \int_{-2}^{4} \left(\int_{x^2/4}^{\frac{x+4}{2}} 1 \, dy \right) dx =$$

$$= \int_{-2}^{4} \left(\frac{x+4}{2} - \frac{x^2}{4} \right) dx = \left[\frac{x^2}{4} + 2x - \frac{x^3}{12} \right]_{x=-2}^{x=4} =$$

$$= 4 + 8 - \frac{16}{3} - \left(1 - 4 + \frac{2}{3} \right) = 15 - \frac{18}{3} = \frac{27}{3} = 9$$

10.5.a Ω: volumen limitado por $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ y planos coordenados



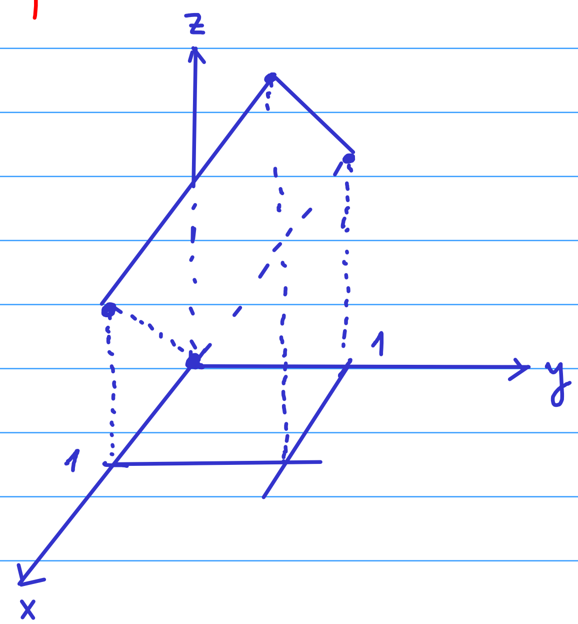
$$V = \iint_{\Omega_p} 4\left(1 - \frac{x}{2} - \frac{y}{3}\right) dx\,dy = 4\int_0^2 \left(\int_0^{3\left(1-\frac{x}{2}\right)} \left(1 - \frac{x}{2} - \frac{y}{3}\right) dy \right) dx$$

$$= 4\int_0^2 \left[\left(1-\frac{x}{2}\right) 3\left(1-\frac{x}{2}\right) - \frac{1}{6}\, 9\left(1-\frac{x}{2}\right)^2\right] dx$$

$$= 4\left[-3\,\frac{\left(1-x/2\right)^2}{2} + \frac{3}{2}\,\frac{\left(1-x/2\right)^3}{3}\right]_{x=0}^{x=2} =$$

$$= 4\left[0 - \left(\frac{-3}{2} + \frac{1}{2}\right)\right] = 4$$

10.5. b) Ω: tronco limitado superiormente por $z = 2x+3y$ e inferiormente por $[0,1]\times[0,1]$

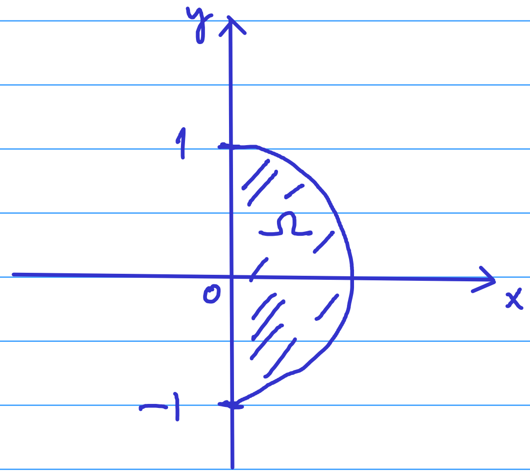


$$V = \iint_{[0,1]\times[0,1]} 2x+3y \, dx\,dy = \int_0^1 \left(\int_0^1 2x+3y \, dx \right) dy =$$

$$= \int_0^1 \left(3y + \left[x^2\right]_{x=0}^{x=1} \right) dy = 3\,\frac{1}{2} + 1 = \frac{5}{2}$$

10.6.a $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx\, dy = I$

Recinto en el que estamos integrando:



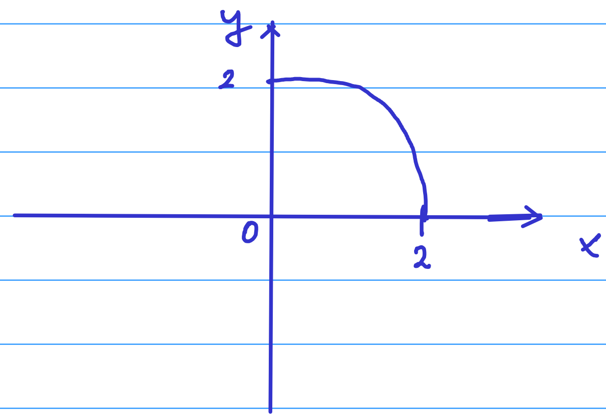
El recinto en coordenadas polares:

$$\Omega_p = \left\{ (r,\theta),\ \frac{-\pi}{2} < \theta < \frac{\pi}{2},\ 0 \leq r < 1 \right\}$$

$$I = \int_0^1 \left(\int_{-\pi/2}^{\pi/2} r\, r \right) d\theta\, dr = \int_0^1 r^2 \pi\, dr = \frac{\pi}{3}$$

10.6.b

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$



$$\Omega_p = \{(r,\theta) : 0 < \theta < \frac{\pi}{2} ,\ 0 < r < 2\}$$

$$I = \int_0^{\pi/2} \int_0^2 r^2\, dr\, d\theta = \frac{8}{3} \cdot \frac{\pi}{2} = \frac{4}{3}\pi$$

10.7 $\Omega = [0,1]\times[0,3]\times[-1,1]$

a) $\iiint_\Omega xyz\,dx\,dy\,dz = \int_0^1\int_0^3\int_{-1}^1 xyz\,dz\,dy\,dx =$

$= \int_0^1\int_0^3 xy\left(\frac{1}{2}-\frac{1}{2}\right)dy\,dx = \int_0^1\int_0^3 0\,dy\,dx = 0$

b) $\iiint_\Omega xe^{y+z}\,dx\,dy\,dz = \int_{-1}^1\int_0^1\int_0^3 xe^{y+z}\,dy\,dx\,dz =$

$= \int_{-1}^1\int_0^1 x\left[e^{y+z}\right]_{y=0}^{y=3} dx\,dz = \int_{-1}^1 \left(e^{3+z}-e^z\right)\left[\frac{x^2}{2}\right]_{x=0}^{x=1} dz$

$= \frac{1}{2}\int_{-1}^1 (e^{3+z}-e^z)\,dz = \frac{1}{2}\left[e^{3+z}-e^z\right]_{z=-1}^{z=1} = \frac{1}{2}\left(e^4-e^1-e^2+e^{-1}\right)$

c) $\iiint_\Omega y^2z^3 \operatorname{sen}x\,dx\,dy\,dz = \int_0^3\int_0^1\int_{-1}^1 y^2z^3\operatorname{sen}x\,dz\,dx\,dy =$

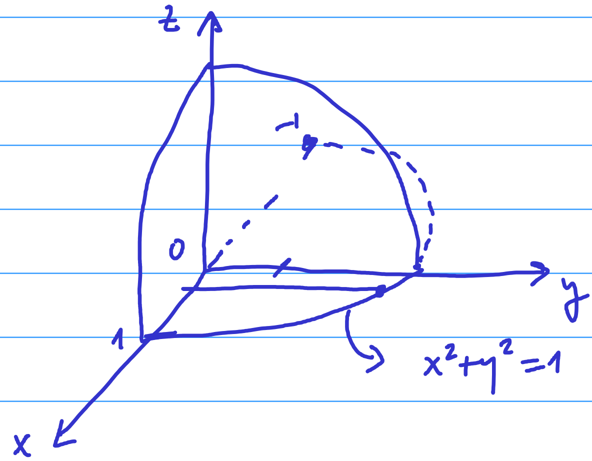
$\int_0^3 y^2\left[\int_0^1 \operatorname{sen}x\left(\int_{-1}^1 z^3\,dz\right)dx\right]dy = \underbrace{\int_{-1}^1 z^3\,dz}_{=0}\cdot\int_0^1\operatorname{sen}x\,dx\cdot\int_0^3 y^2\,dy$

$= 0$

10.8.a $\Omega = \{(x,y,z) \in \mathbb{R}^3 \text{ tq } x^2+y^2+z^2=1\}$

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = I$$

$\Omega = \{(x,y,z) : -1 \le x \le 1, -\sqrt{1-x^2} \le y \le +\sqrt{1-x^2}, -\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}\}$



$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \iiint_\Omega y^3+z+x\,dx\,dy\,dz =$$

$$= \iiint_\Omega y^3\,dx\,dy\,dz + \iiint_\Omega z\,dx\,dy\,dz + \iiint_\Omega x\,dx\,dy\,dz = 0+0+0=0$$

10.8.b) $\iiint_\Omega (y \operatorname{sen} z + x)\, dx\, dy\, dz = I$

$\Omega = \{(x,y,z) : y \geq z \geq y^2,\ 0 \leq x, y \leq 1\}$

$$I = \int_0^1 \int_0^1 \int_y^{y^2} (y \operatorname{sen} z + x)\, dz\, dx\, dy =$$

$$= \int_0^1 \int_0^1 \left[x(y^2 - y) + y(\cos y - \cos y^2) \right] dx\, dy =$$

$$= \int_0^1 \frac{1}{2}(y^2 - 1) + y(\cos y - \cos y^2)\, dy =$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - y\right]_{y=0}^{y=1} + \int_0^1 y \cos y\, dy - \frac{1}{2}\int_0^1 2y \cos y^2\, dy =$$

$$= \frac{1}{2}\left(\frac{1}{3} - 1\right) - \frac{1}{2}\left[\operatorname{sen} y^2\right]_{y=0}^{y=1} + \int_0^1 y \cos y\, dy =$$

$$= \frac{-1}{3} - \frac{\operatorname{sen} 1}{2} + \int_0^1 y \cos y\, dy = (**)$$

$$\int y \cos y\, dy = y \operatorname{sen} y - \int \operatorname{sen} y\, dy = y \operatorname{sen} y + \cos y$$

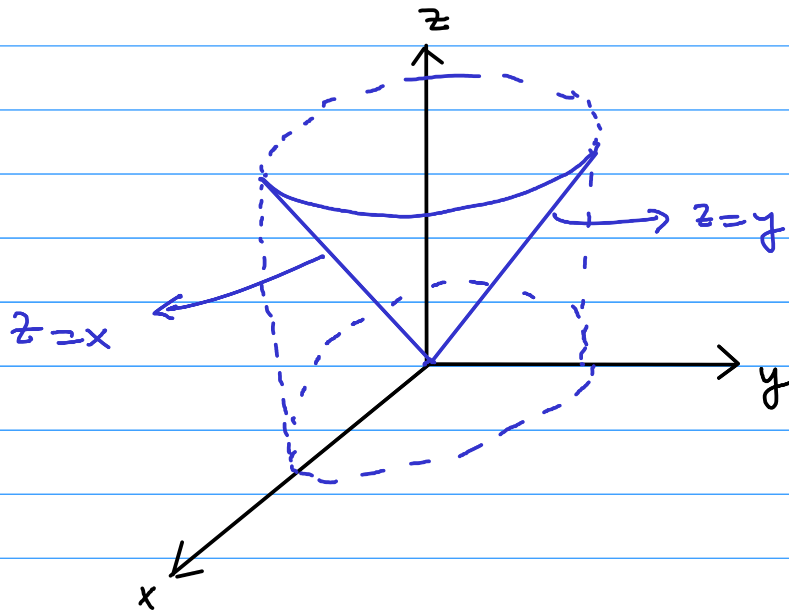
$y = u \qquad \cos y\, dy = dv$

$dy = du \qquad \operatorname{sen} y = v$

$$(**) = \frac{-1}{3} - \frac{\operatorname{sen} 1}{2} + 1 \operatorname{sen} 1 + \cos 1 - \cos 0 =$$

$$= \frac{-4}{3} + \frac{1}{2} \operatorname{sen} 1 + \cos 1$$

10.9



$$\left.\begin{array}{l} z=\sqrt{x^2+y^2} \\ z=1 \end{array}\right\} \Rightarrow x^2+y^2=1$$

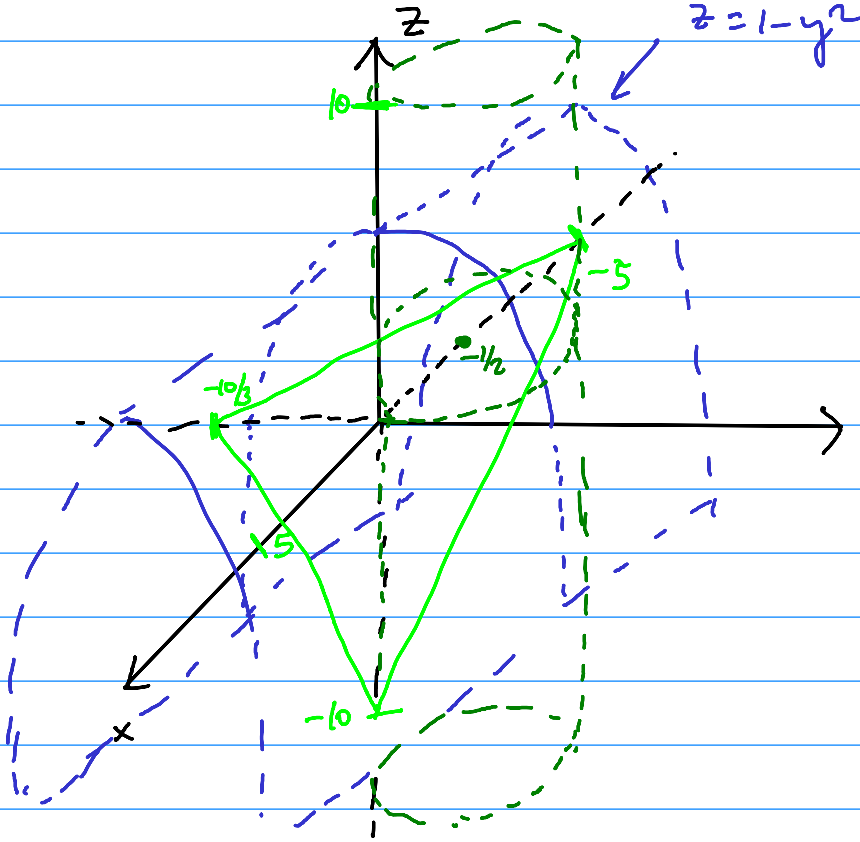
$\Omega = \{ (x,y) \in \mathbb{R}^2 \text{ tq } x^2+y^2 \leq 1 \}$

$$V = \iint_\Omega 1-\sqrt{x^2+y^2}\, dx\,dy = \int_0^1 \int_0^{2\pi} (1-r)\, r \, d\theta\, dr =$$

$$= 2\pi \int_0^1 (1-r)\, r\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^3}{3} \right]_{r=0}^{r=1} = 2\pi \left(\frac{1}{2} - \frac{1}{3} \right) =$$

$$= 2\pi \, \frac{1}{6} = \frac{\pi}{3}$$

10.10



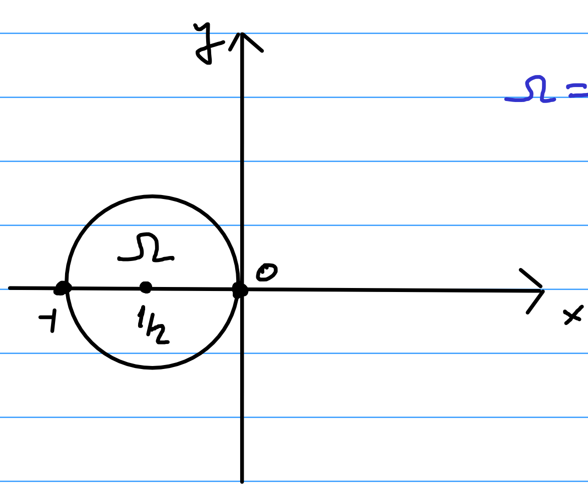
$2x+3y+z+10=0$

$$x^2+y^2+x=0$$
$$\Updownarrow$$
$$\left(x+\frac{1}{2}\right)^2+y^2=\frac{1}{4}$$

$$\Omega=\left\{(x,y) \text{ tq } \left(x+\frac{1}{2}\right)^2+y^2\leq\frac{1}{4}\right\}$$

$$V=\iint_\Omega 1-y^2-(-10-2x-3y)\,dx\,dy$$

Describimos el conjunto Ω:



$$\Omega=\left\{(x,y):\ 0\leq x\leq 0,\ -\sqrt{\frac{1}{4}-\left(x+\frac{1}{2}\right)^2}\leq y \leq +\underbrace{\sqrt{\frac{1}{4}-\left(x+\frac{1}{2}\right)^2}}_{A}\right\}$$

$$V=\iint_{\Omega} 1-y^2-(-10-2x-3y)\,dx\,dy=$$

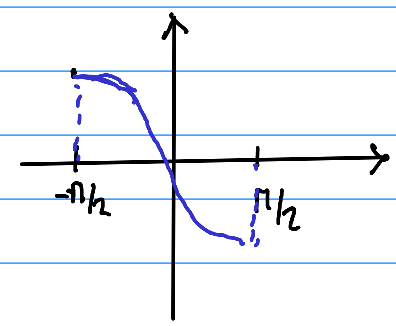
$$=\int_{-1}^{0}\int_{-A}^{+A} 1-y^2+10+2x+3y\;dy\;dx \;=$$

$$=\int_{-1}^{0} 22A+4Ax-\frac{1}{3}\,2A^3\;dx \overset{*}{=}$$

$$\left\{\begin{array}{l} x+\frac{1}{2}=\frac{1}{2}\operatorname{sen}\theta \qquad \frac{1}{4}-\left(x+\frac{1}{2}\right)^2=\frac{1}{4}\cos^2\theta \\ x\in[-1,0] \Rightarrow x+\frac{1}{2}\in\left[-\frac{1}{2},\frac{1}{2}\right] \Rightarrow \theta\in\left[-\pi/2,\pi/2\right] \\ dx=\frac{1}{2}\cos\theta\,d\theta \end{array}\right.$$

$$\overset{*}{=}\int_{-\pi/2}^{\pi/2}\left[22\,\frac{1}{2}\cos\theta+4\,\frac{1}{2}\cos\theta\left(\frac{1}{2}\operatorname{sen}\theta-\frac{1}{2}\right)-\frac{2}{3}\,\frac{1}{8}\cos^3\theta\right]\frac{1}{2}\cos\theta\,d\theta \;\overset{**}{=}$$

Observa que $\int_{-\pi/2}^{\pi/2}\operatorname{sen}\theta\,d\theta=\int_{-\pi/2}^{\pi/2}\cos^2\theta\operatorname{sen}\theta\,d\theta=0$ porque ambas funciones son impares.



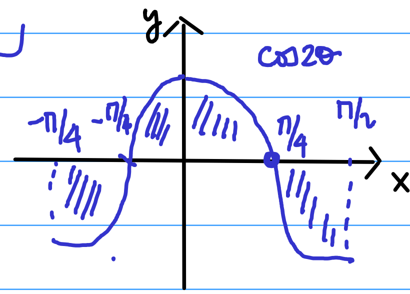
$$\overset{**}{=}\frac{10}{2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta-\frac{1}{24}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta \overset{**}{=}$$

$$\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{1}{2}\cdot\pi+\frac{1}{2}\left[\frac{\operatorname{sen}2\theta}{2}\right]_{\theta=-\pi/2}^{\theta=\pi/2}=$$

$$=\frac{\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} 1 + \cos^2(2\theta) + 2\cos(2\theta) \, d\theta =$$

$$\frac{\pi}{4} + \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2(2\theta)\, d\theta + \underbrace{\frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos(2\theta)\, d\theta}_{=0} =$$

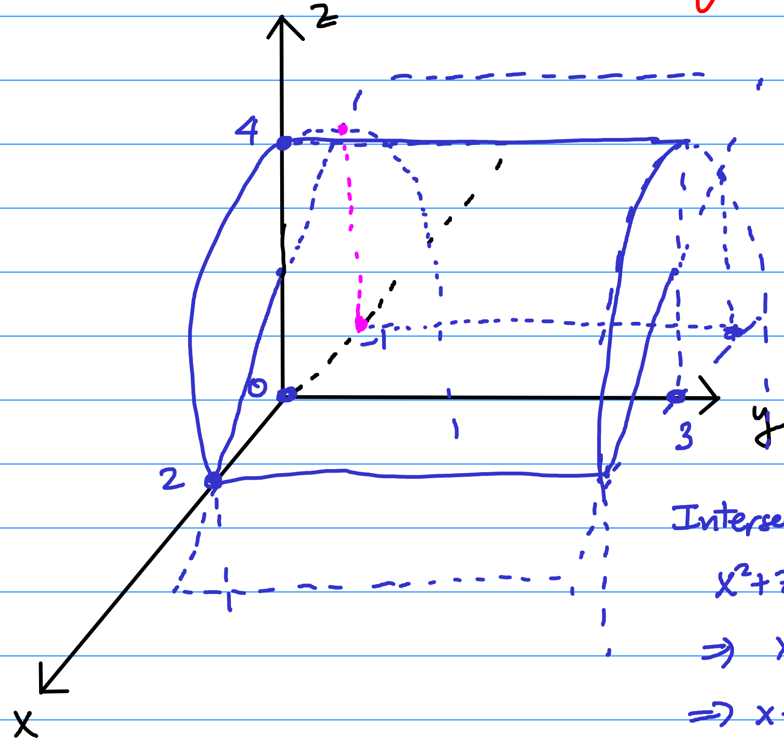


$$= \frac{\pi}{4} + \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1+\cos 4\theta}{2} \, d\theta = \frac{\pi}{4} + \frac{\pi}{8} + \frac{1}{8}\int_{-\pi/2}^{\pi/2} \frac{\cos(4\theta)}{2} \, d\theta = \frac{\pi}{4} + \frac{\pi}{8}$$

(***)

$$= \frac{5}{2}\pi - \frac{1}{24} \cdot \left(\frac{\pi}{4} + \frac{\pi}{8}\right) = \frac{5}{2}\pi - \frac{1}{96}\pi - \frac{1}{192}\pi = \frac{159}{64}\pi$$

10.12 Volumen limitado superiormente por $x^2+z=4 \iff z=4-x^2$
inferiormente por $x+z=2$
lateralmente por $y=0$ e $y=3$



Intersección de $x+z=2$ con $x^2+z=4 \Rightarrow x^2-x+2=4 \Rightarrow$
$\Rightarrow x^2-x-2=0 \Rightarrow$
$\Rightarrow x=\dfrac{1\pm\sqrt{1+8}}{2}=\dfrac{1\pm 3}{2} \begin{cases} 2 \\ -1 \end{cases}$

Queremos calcular el volumen de

$$\Omega=\{(x,y,z) \text{ tq } 0\leq y\leq 3,\ -1\leq x\leq 2,\ 2-x\leq z\leq 4-x^2\}$$

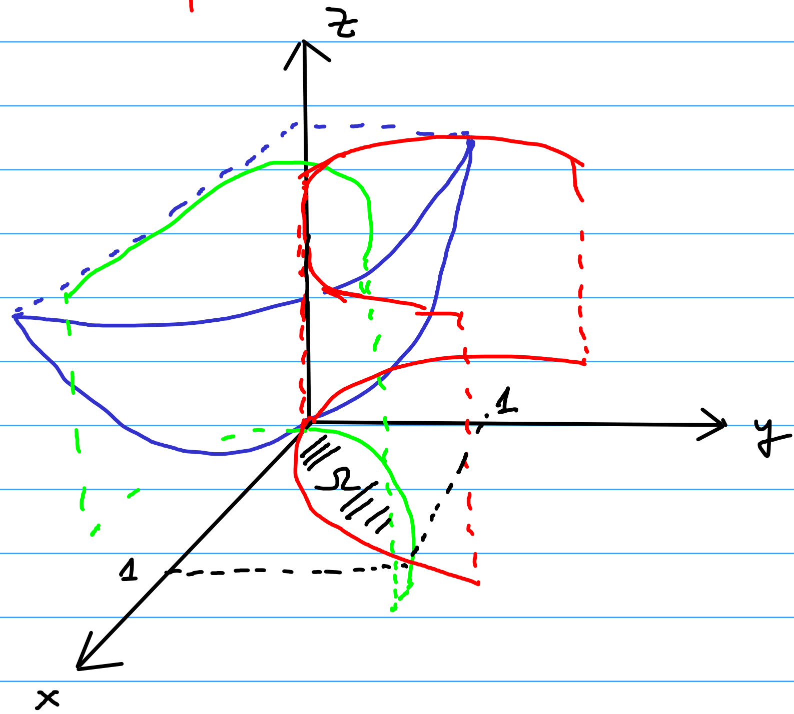
$$V=\iiint_\Omega 1\,dx\,dy\,dz=\int_0^3\int_{-1}^2\int_{2-x}^{4-x^2} 1\ dz\ dx\ dy=$$

$$=\int_0^3\int_{-1}^2(4-x^2-2+x)dx\ dy=3\cdot\int_{-1}^2(2-x^2+x)dx=3\left[2x-\frac{x^3}{3}+\frac{x^2}{2}\right]_{x=-1}^{x=2}$$

$$=3\cdot\left[\left(4-\frac{8}{3}+2\right)-\left(-2-\frac{1}{3}+\frac{1}{2}\right)\right]=3\left(6-\frac{7}{3}+\frac{3}{2}\right)=$$

$$=3\left(\frac{36-14+9}{6}\right)=\frac{31}{2}$$

10.14 Volumen limitado por $z=x^2+4y^2$, $z=0$, $x=y^2$, $x^2=y$



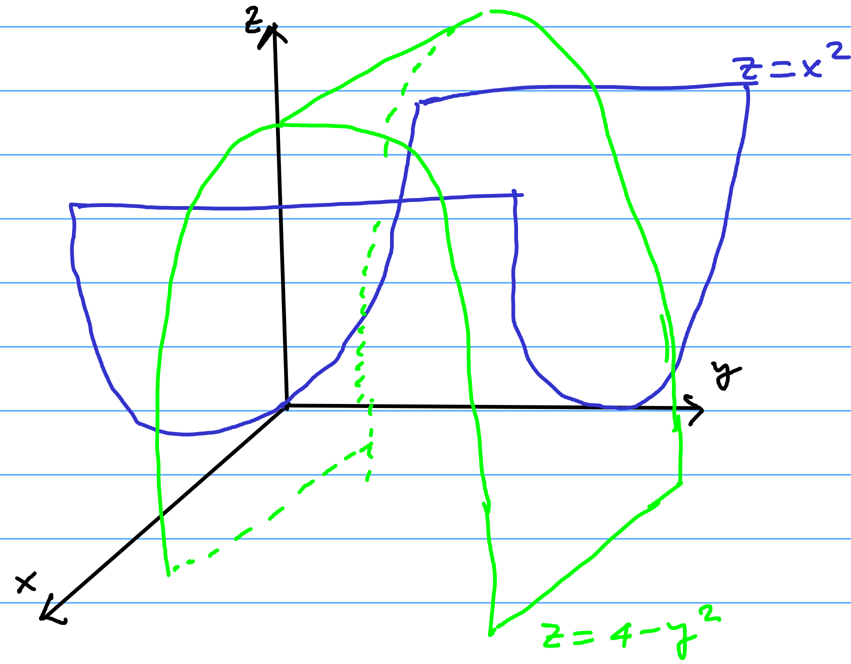
$$V=\iint_{\Omega} x^2+4y^2 \, dx\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+4y^2)\, dy\, dx$$

$$=\int_0^1 x^2(\sqrt{x}-x^2)+\frac{4}{3}\left(x^{3/2}-x^6\right) dx =$$

$$=\int_0^1 x^{5/2}-x^4+\frac{4}{3}x^{3/2}-\frac{4}{3}x^6 \, dx =$$

$$=\frac{2}{7}-\frac{1}{5}+\frac{4}{3}\,\frac{2}{5}-\frac{4}{3}\,\frac{1}{7} = \frac{2}{7}-\frac{1}{5}-\frac{8}{15}-\frac{4}{21}$$

10.16



Los cilindros intersecan en $4-y^2=x^2 \Rightarrow x^2+y^2=4$. Así que si $\Omega = \{(x,y) \text{ tq } x^2+y^2=4\}$ entonces:

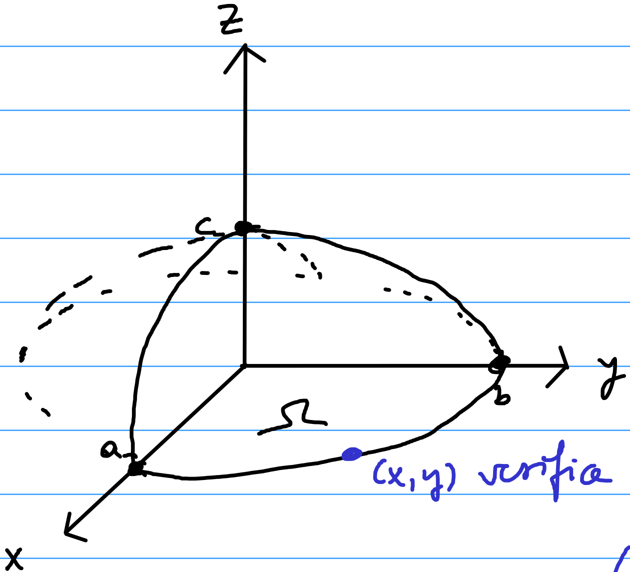
$$V=\iint_\Omega 4-y^2-x^2 \, dx\,dy = \iint_\Omega 4-(x^2+y^2)\, dx\,dy =$$

$$= \int_0^2 \int_0^{2\pi} (4-r^2)\, r \, d\theta\, dr = 2\pi \int_0^2 (4-r^2)\, r\, dr =$$

$$= 2\pi \left(4\left[\frac{r^2}{2}\right]_{r=0}^{r=2} - \left[\frac{r^4}{4}\right]_{r=0}^{r=2} \right) = 2\pi \left(2\cdot 4 - \frac{1}{4} 16 \right)$$

$$= 2\pi\,(8-4) = 8\pi$$

(10.17) Calcular el volumen de $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



(x,y) verifica $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ luego $\left(\frac{x}{a},\frac{y}{b}\right)$ le corresponde $r=1$ en la nuevas coorden.

$$V=\iint_\Omega 2\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dx\,dy$$

$$\Omega=\left\{(x,y)\ \text{tq}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\right\}$$

Planteamos el siguiente cambio de coordenadas:

$\phi(r,\theta)=(ar\cos\theta,\ br\,\text{sen}\,\theta)$ definido en $(0,+\infty)\times(0,2\pi)$ del que se puede ver fácilmente que es biyectivo y que tiene jacobiano:

$$J\phi(r,\theta)=\begin{pmatrix} a\cos\theta & -ar\,\text{sen}\,\theta \\ b\sin\theta & br\cos\theta \end{pmatrix}$$

$$\left|\det J\phi(r,\theta)\right| = abr \neq 0.$$

Para calcular las coordenadas "polares" de un punto (x,y)

basta con tomar las polares del punto $\left(\frac{x}{a}, \frac{y}{b}\right)$.

El recinto Ω descrito en estas nuevas coordenadas es:

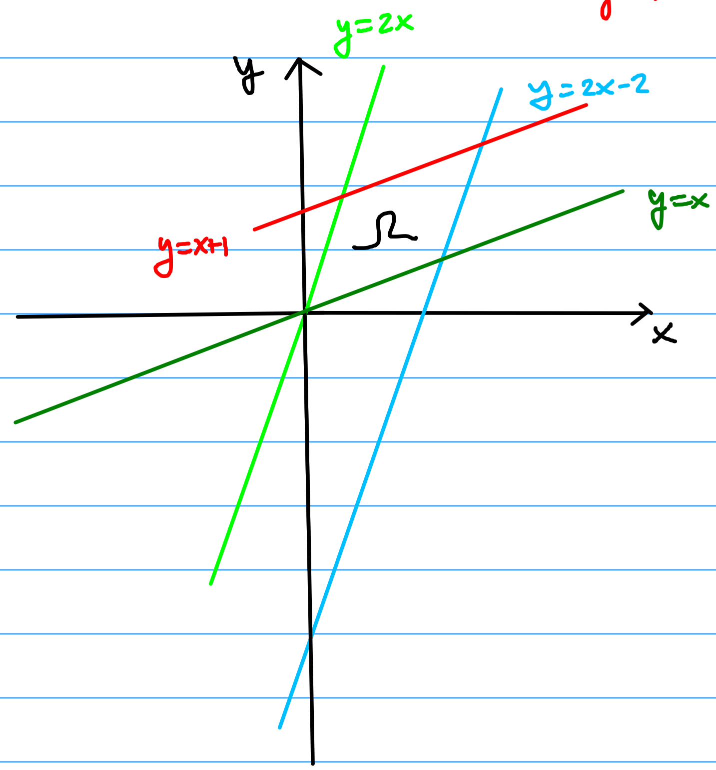
$$\Omega_p = \{(r,\theta) : 0 \leq r < 1 , 0 < \theta < 2\pi\}$$

$$V = \iint_\Omega 2\sqrt{c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\, dx\,dy = \int_0^1 \int_0^{2\pi} 2\sqrt{1-r^2}\; cabr\, d\theta\, dr$$

$$= -abc \int_0^1 \int_0^{2\pi} -2r\sqrt{1-r^2}\, d\theta\, dr = -abc\, 2\pi\, \frac{2}{3}\left[1-r^2\right]_{r=0}^{r=1}$$

$$= -\frac{4}{3} abc\,\pi\,(0-1) = \frac{4}{3} abc\,\pi.$$

(10.18) $\iint_\Omega xy\,dx\,dy$ $\quad \Omega$ limitado por $y=2x$, $y=2x-2$, $y=x$ e $y=x+1$



Planteamos el cambio de variable $x=u-v$, $y=2u-v$.

$$\phi:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(u,v) \longrightarrow (u-v,\ 2u-v)$$

$$|J\phi(u,v)| = \left|\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\right| = -1+2 = 1$$

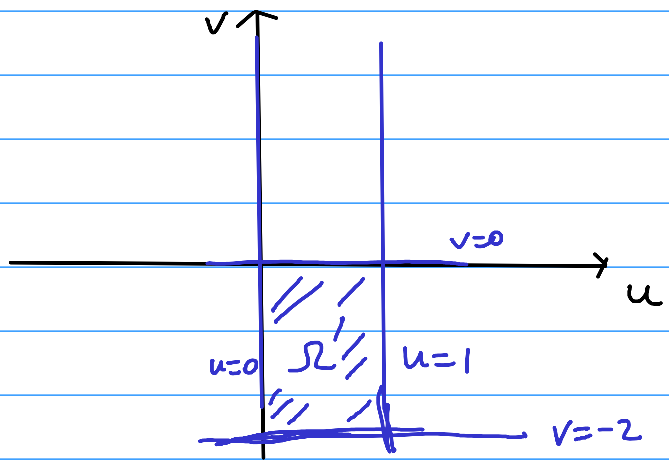
ϕ tiene inversa. $\quad \phi(u,v)=(x,y) \Rightarrow \begin{cases} x=u-v \\ y=2u-v \end{cases}$

$\Rightarrow \quad y-x=u, \quad y-2x=v$

Transformamos el recinto Ω a las nuevas coordenadas:

$$y = x \to 2u - v = u - v \Rightarrow u = 0$$
$$y = 2x \to \cancel{2u} - v = \cancel{2u} - 2v \Rightarrow v = 0$$
$$y = x+1 \quad 2u - \cancel{v} = u - \cancel{v} + 1 \Rightarrow u = 1$$
$$y = 2x - 2 \quad \cancel{2u} - v = \cancel{2u} - 2v - 2 \Rightarrow v = -2$$

El recinto se ha transformado en Ω':



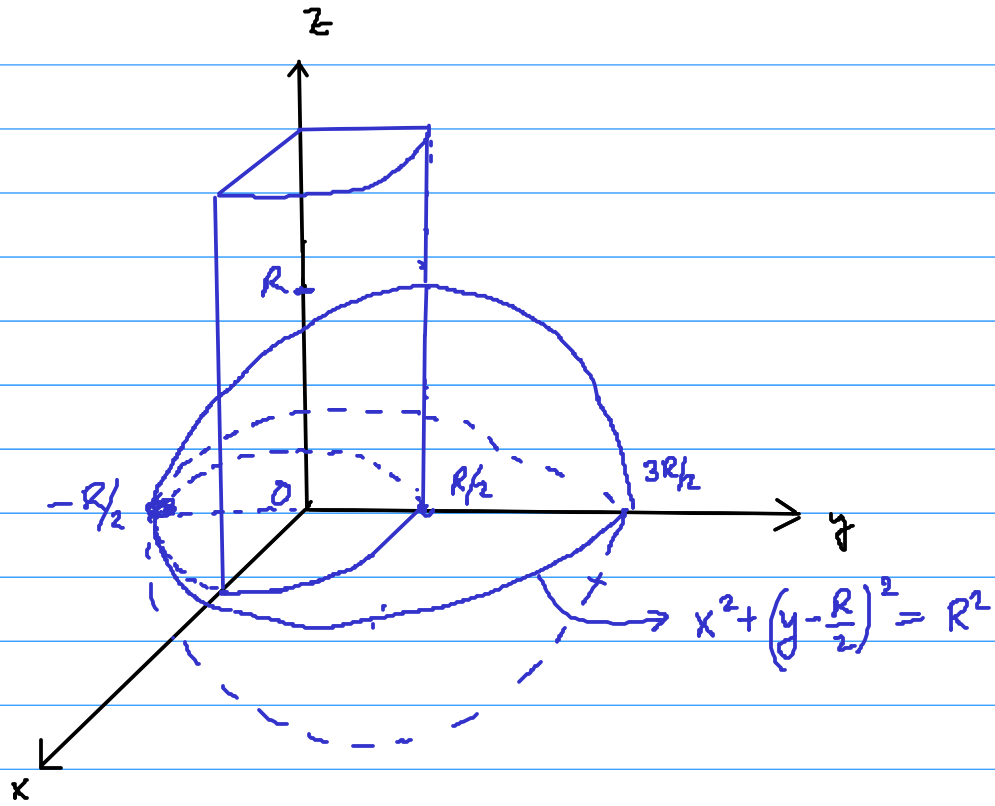
$$I = \iint_\Omega xy\,dx\,dy = \iint_{\Omega'} (u-v)(2u-v) \overset{J\phi}{1}\, du\,dv =$$

$$= \int_0^1 \int_{-2}^0 (u-v)(2u-v)\,dv\,du = \int_0^1 \int_{-2}^0 2u^2 - uv - 2uv + v^2\,dv\,du$$

$$= \int_0^1 4u^2 - 3u\left(0 - \frac{4}{2}\right) + \left(0 - \frac{-8}{3}\right) du =$$

$$= \frac{4}{3} + 6\,\frac{1}{2} + \frac{8}{3} = \frac{12}{3} + 3 = \frac{21}{3} = 7$$

10.19



Ecuación de la esfera: $x^2+\left(y-\frac{R}{2}\right)^2+z^2=R^2$

Ecuación del cilindro: $x^2+y^2=\frac{R^2}{4}$

Los círculos intersección de estas superficies con $z=0$ son
$x^2+y^2=R^2/4$ y $x^2+(y-R/2)^2=R^2$

Éstos intersecan en:

$$\frac{R^2}{4}-y^2+\left(y-\frac{R}{2}\right)^2=R^2 \Rightarrow$$

$$\Rightarrow \frac{R^2}{4}-\cancel{y^2}+\cancel{y^2}-Ry+\frac{R^2}{4}=R^2 \Rightarrow \quad Ry=-\frac{1}{2}R^2$$

$$\Rightarrow y=-\frac{1}{2}R$$

Esto garantiza que el círculo que da el cilindro está contenido dentro del que da la esfera tal y como lo hemos dibujado en la figura.

Llamamos $\Omega = \{(x,y) \text{ tq } x^2+y^2 \leq R^2/4\}$

$$V = \iint_\Omega \sqrt{R^2 - x^2 - \left(y - \frac{R}{2}\right)^2} - \left(-\sqrt{R^2 - x^2 - \left(y - \frac{R}{2}\right)^2}\right) dx\, dy$$

$$= 2\iint_\Omega \sqrt{R^2 - x^2 - \left(y - \frac{R}{2}\right)^2}\, dx\, dy =$$

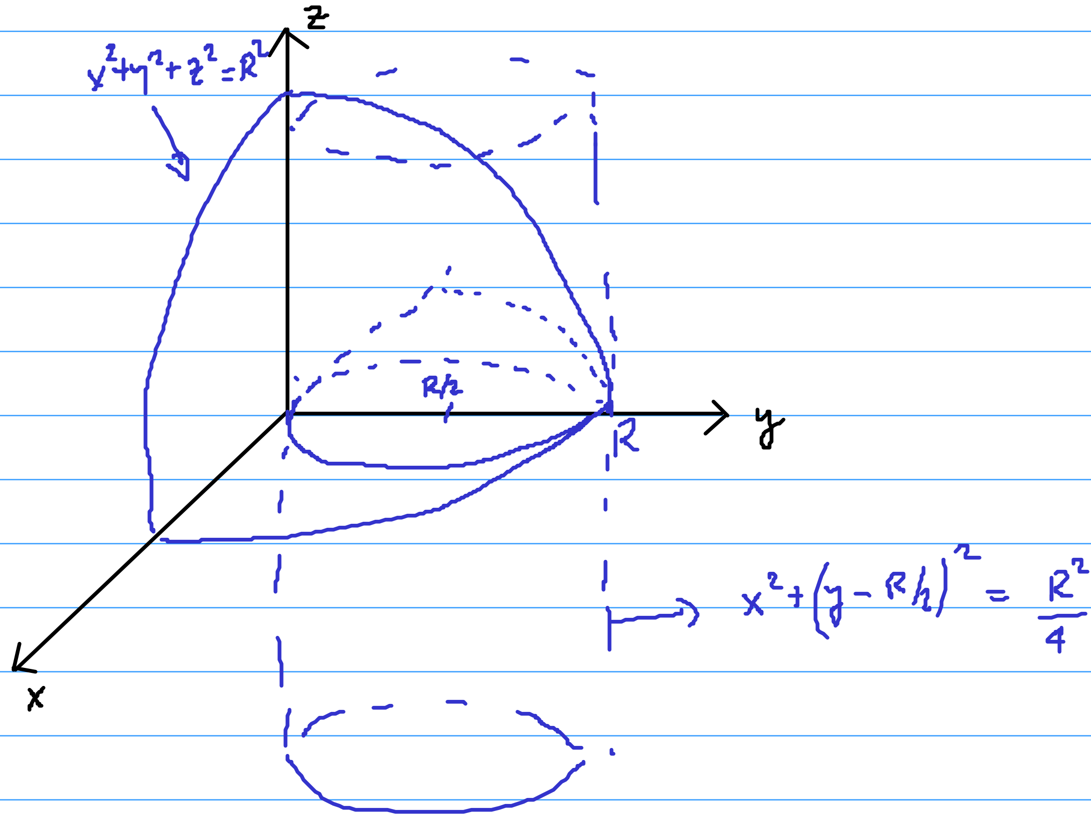
$$= 2\int_0^{R/2} \int_0^{2\pi} \sqrt{R^2 - r^2\cos^2\theta - \left(r\,\mathrm{sen}\,\theta - \frac{R}{2}\right)^2}\; r\, d\theta\, dr$$

$$= 2\int_0^{R/2} \int_0^{2\pi} r\sqrt{R^2 \underbrace{- r^2\cos^2\theta}_{} \underbrace{- r^2\,\mathrm{sen}^2\theta}_{} - \frac{R^2}{4} + Rr\,\mathrm{sen}\,\theta}\; d\theta\, dr$$

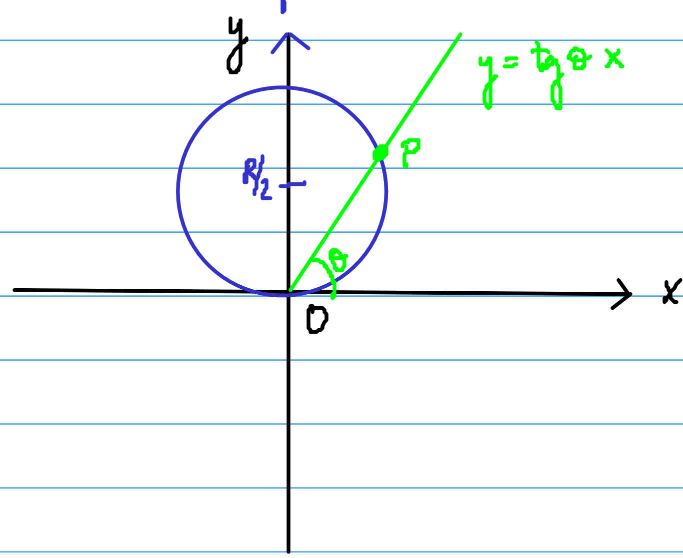
$$= 2\int_0^{R/2} r \int_0^{2\pi} \sqrt{\frac{3}{4}R^2 - r^2 + Rr\,\mathrm{sen}\,\theta}\; d\theta\, dr$$

(estos cálculos son muy complicados)

Probamos haciendo el dibujo del problema de otra forma



Sea Ω el círculo $\{(x,y)$ tq $x^2+(y-R/2)^2 \leq R^2/4\}$, describámoslo en coordenadas polares:



<u>Cálculo del punto $P(x,y)$</u>

$$\left\{\begin{array}{l} y = tg\theta\, x \\ x^2+\left(y-\frac{R}{2}\right)^2 = \frac{R^2}{4} \end{array}\right\} \Rightarrow x^2+x^2tg^2\theta - R\,tg\theta\, x + \frac{R^2}{4} = R^2/4$$

$$\Rightarrow \quad x^2(1+tg^2\theta) - R\,tg\,\theta\, x = 0 \Rightarrow$$

$$x\left[x(1+tg^2\theta) - R\,tg\,\theta\right] = 0 \Rightarrow \begin{cases} x=0 \\ x = \dfrac{R\,tg\,\theta}{1+tg^2\theta} \end{cases}$$

$$P = \left(\frac{R\,tg\,\theta}{1+tg^2\theta} , \frac{R\,tg^2\theta}{1+tg^2\theta} \right)$$

$$d(P,O) = \sqrt{\frac{R^2 tg^2\theta + R^2 tg^4\theta}{(1+tg^2\theta)^2}} = \frac{R\,tg\,\theta}{1+tg^2\theta}\sqrt{1+tg^2\theta} =$$

$$= \frac{R\,tg\,\theta}{\sqrt{1+tg^2\theta}}$$

$$\Omega_p = \left\{ (r,\theta) : 0<\theta<\pi , \quad 0<r< \frac{R\,tg\,\theta}{\sqrt{1+tg^2\theta}} \right\}$$

$$V = \iint_\Omega \sqrt{R^2-x^2-y^2} - \left(-\sqrt{R^2-x^2-y^2}\right) dx\,dy =$$

$$= \iint_{\Omega_p} 2\sqrt{R^2-r^2}\; r\; dr\,d\theta = -\int_0^\pi \int_0^{\frac{R\,tg\,\theta}{\sqrt{1+tg^2\theta}}} -2\sqrt{R^2-r^2}\; r\,dr\,d\theta$$

$$-\int_0^\pi \left[\frac{2}{3}(R^2-r^2)^{3/2}\right]_{r=0}^{r=\frac{R\,tg\,\theta}{\sqrt{1+tg^2\theta}}} d\theta =$$

$$= -\frac{2}{3}\int_0^{\pi} \left(R^2 - \frac{R^2 tg^2\theta}{1+tg^2\theta}\right)^{3/2} - R^3 \, d\theta$$

$$= -\frac{2}{3}\int_0^{\pi} \left(\frac{R^2}{1+tg^2\theta}\right)^{3/2} - R^3 \, d\theta = \frac{-2}{3}\int_0^{\pi} \left(R^2 \cos^2\theta\right)^{3/2} - R^3 \, d\theta$$

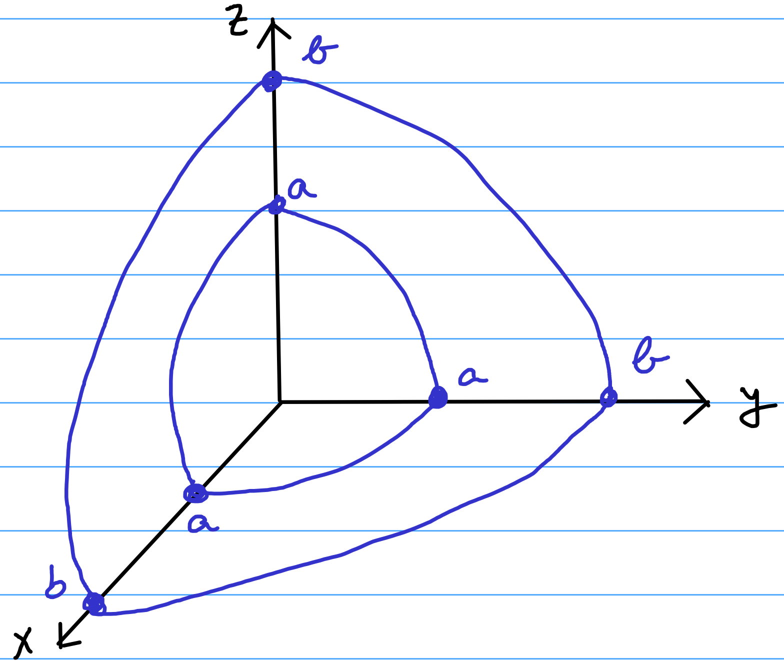
$$= \frac{-2}{3}\int_0^{\pi} \left(R^3 \cos^3\theta - R^3\right) d\theta = \frac{2}{3}\pi R^3$$

10.20

$$\iiint_\Omega \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}} = I$$

Ω región limitada por las esferas

$x^2+y^2+z^2=a^2$ y $x^2+y^2+z^2=b^2$ $\quad 0<a<b$



$\Omega_e = \{(r,\theta,\varphi) : a<r<b,\ 0<\theta<2\pi,\ 0<\varphi,\pi\}$

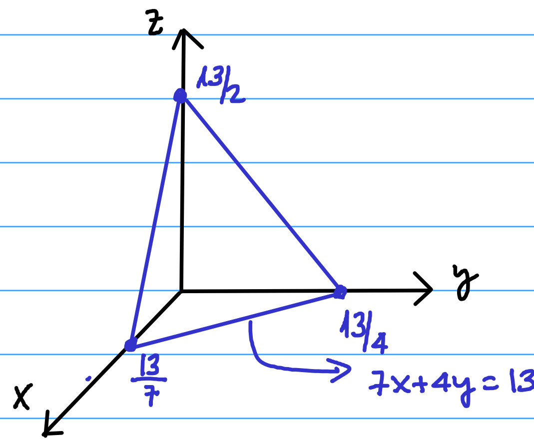
$$I = \int_a^b \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^3}\, r^2 \operatorname{sen}\varphi \, d\varphi\, d\theta\, dr =$$

$$= \int_a^b \frac{1}{r} \int_0^{2\pi} \left[-\cos\varphi\right]_{\varphi=0}^{\varphi=\pi} d\theta\, dr = 2\pi \cdot (\cos 0 - \cos\pi) \int_a^b \frac{1}{r}\, dr$$

$$= 4\pi \cdot (\log b - \log a) = 4\pi \log\left(\frac{b}{a}\right).$$

(10.27) $\iiint_\Omega 4xy^9 + e^{5z}\, dx\,dy\,dz = I$

Ω es el tetraedro limitado por $x=0,\ y=0,\ z=0$ y $7x+4y+2z=13$



$$I = \int_0^{13/7}\int_0^{\frac{13-7x}{4}}\int_0^{\frac{13-7x-4y}{2}} yz \; dz\,dy\,dx =$$

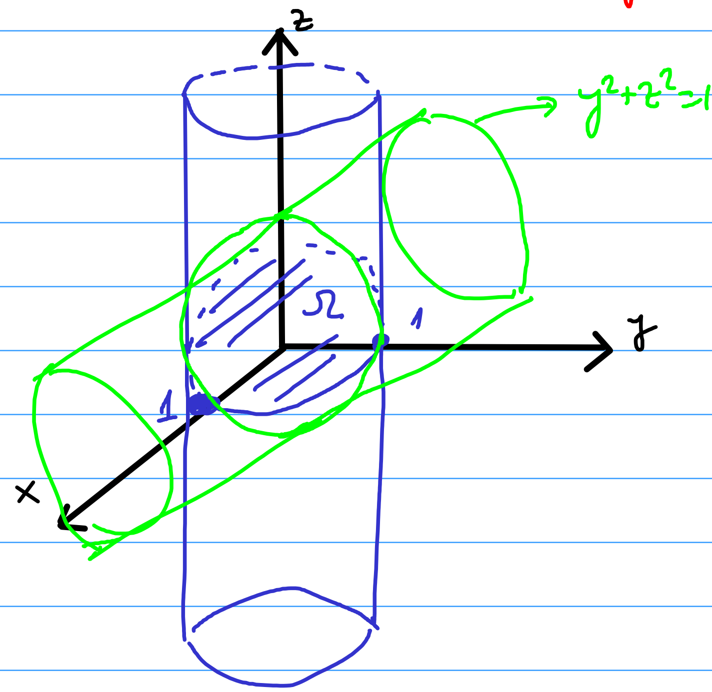
$$= \int_0^{13/7}\int_0^{\frac{13-7x}{4}} y\,\frac{13-7x-4y}{2}\, dy\,dx$$

$$= \frac{1}{2}\int_0^{13/7} \left[13\frac{y^2}{2} - 7x\frac{y^2}{2} - 4\frac{y^3}{3}\right]_{y=0}^{\frac{13-7x}{4}} dx$$

$$= \frac{1}{2}\int_0^{13/2} \frac{13}{2}\left(\frac{13-7x}{4}\right)^2 - \frac{7}{2}x\left(\frac{13-7x}{4}\right)^2 - \frac{4}{3}\left(\frac{13-7x}{4}\right)^3 dx = \dots$$

10.28 Calcule el volumen del conjunto

$$\Omega = \{(x,y,z) : x^2+y^2 \le 1,\ y^2+z^2 \le 1\}$$



$$V(\Omega) = \iint_{\Omega} \sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\, dx\,dy = \iint_{\Omega} \sqrt{1-y^2}\, dx\,dy =$$

$$= \int_0^{2\pi}\int_0^1 2\sqrt{1-r^2 \operatorname{sen}^2\theta}\ r\ dr\, d\theta =$$

$$= -\int_0^{2\pi}\int_0^1 \frac{-2\operatorname{sen}^2\theta}{\operatorname{sen}^2\theta}\sqrt{1-r^2\operatorname{sen}^2\theta}\ r\ dr\ d\theta =$$

$$-\int_0^{2\pi} \frac{1}{\operatorname{sen}^2\theta}\left[\frac{2}{3}\left(1-r^2\operatorname{sen}^2\theta\right)^{3/2}\right]_{r=0}^{r=1} d\theta = -\frac{2}{3}\int_0^{2\pi} \frac{1}{\operatorname{sen}^2\theta}\left[\left(1-\operatorname{sen}^2\theta\right)^{3/2} - 1\right] d\theta$$

$$= -\frac{2}{3}\int_0^{2\pi} \underbrace{\frac{1}{\operatorname{sen}^2\theta}\cos^3\theta}_{f(\theta)} - \frac{1}{\operatorname{sen}^2\theta}\ d\theta$$

PROPIEDADES DE $f(\theta)$

$$f(\pi+\theta) = -f(\theta)$$

Demost.

$$f(\pi+\theta) = \frac{\cos^3(\pi+\theta)}{\operatorname{sen}^2(\pi+\theta)} = \frac{(\cos\pi\cos\theta - \cancel{\operatorname{sen}\pi}\operatorname{sen}\theta)^3}{(\cancel{\operatorname{sen}\pi\cos\theta} + \cos\pi\operatorname{sen}\theta)^2} =$$

$$= \frac{-\cos^3\theta}{\operatorname{sen}^2\theta} = -f(\theta)$$

Consecuencia:

$$\int_0^{\pi} f(\theta)\,d\theta = \int_0^{\pi} -f(\pi+\theta)\,d\theta = \int_{\pi}^{2\pi} -f(u)\,du$$

$$u = \theta+\pi, \quad du = d\theta$$

Así que:

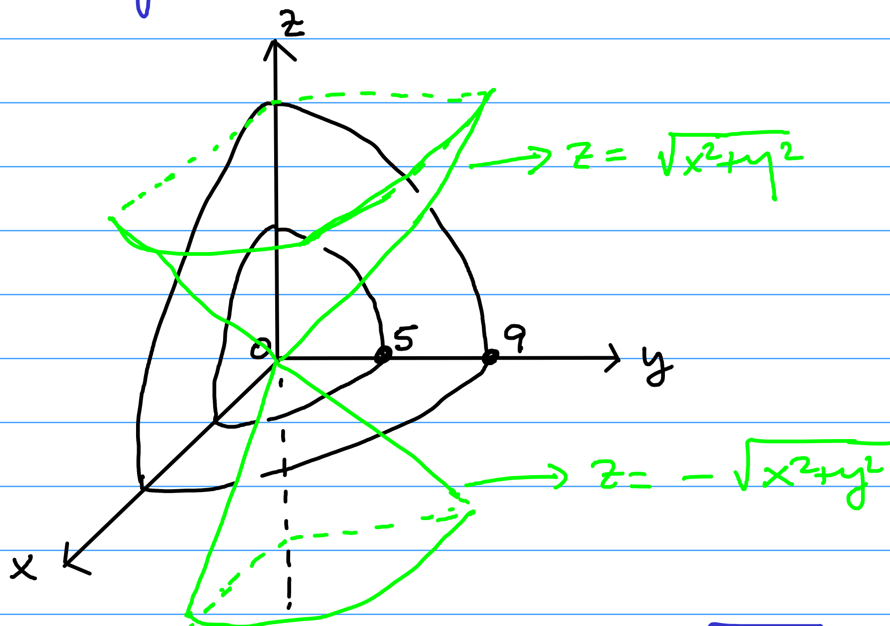
$$\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{\pi} f(\theta)\,d\theta + \int_{\pi}^{2\pi} f(\theta)\,d\theta = 0$$

- $\displaystyle\int \frac{1}{\operatorname{sen}^2\theta}\,d\theta = \int \frac{2}{1-\cos 2\theta}\,d\theta$ .... CONTINUARÁ

(10.29) Calcular el volumen de

$$\Omega = \{ (x,y,z) \in \mathbb{R}^3 \text{ tq } 25 \leq x^2+y^2+z^2 \leq 81, \ z^2 \leq x^2+y^2 \}$$

Describimos el conjunto en coordenadas cilíndricas:



$$z^2 \leq x^2+y^2 \Leftrightarrow |z| \leq \sqrt{x^2+y^2} \Leftrightarrow \begin{cases} z \leq \sqrt{x^2+y^2} & \text{si } z \geq 0 \\ -z \leq \sqrt{x^2+y^2} & \text{si } z < 0 \\ z \geq -\sqrt{x^2+y^2} & \end{cases}$$

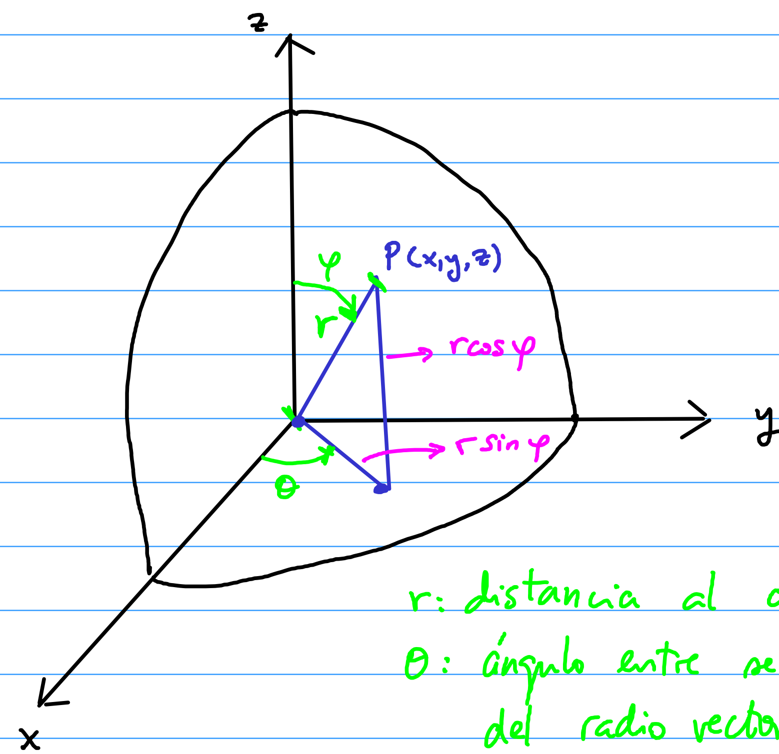
$$\Omega_c = \{ (r,\theta,z) : 5 \leq r \leq 9, 0 < \theta < 2\pi, -r \leq z \leq r \}$$

$$V(\Omega) = \iiint_{\Omega} 1 \, dx\,dy\,dz = \iiint_{\Omega_c} 1 \, r \, dr\,d\theta\,dz =$$

$$= \int_5^9 \int_0^{2\pi} \int_{-r}^{r} r \, dz\,d\theta\,dr = 4\pi \int_5^9 r^2 \, dr = 4\pi \left[\frac{r^3}{3} \right]_{r=5}^{r=9} =$$

$$= \frac{4\pi}{3} (9^3 - 5^3).$$

COORDENADAS ESFÉRICAS



r: distancia al origen

θ: ángulo entre semieje x^+ y proy. del radio vector $\overrightarrow{OP}$

φ: colatitud

$$\phi : (0,+\infty) \times (0,2\pi) \times (0,\pi) \longrightarrow \mathbb{R}^3 \setminus A$$

$$(r, \theta, \varphi) \longrightarrow (x,y,z) = (r\cos\theta \,\text{sen}\varphi,\ r\,\text{sen}\,\theta\,\text{sen}\varphi,\ r\cos\varphi)$$

$$A = \{(x,y,z) \text{ tq } y=0,\ x \geq 0\}$$

$$|J\phi(r,\theta,\varphi)| = \begin{vmatrix} \cos\theta\,\text{sen}\,\varphi & -r\,\text{sen}\theta\,\text{sen}\,\varphi & r\cos\theta\cos\varphi \\ \text{sen}\theta\,\text{sen}\varphi & r\cos\theta\,\text{sen}\varphi & r\,\text{sen}\theta\cos\varphi \\ \cos\varphi & 0 & -r\,\text{sen}\varphi \end{vmatrix} =$$

$$= r^2\,\text{sen}\varphi \begin{vmatrix} \cos\theta\,\text{sen}\varphi & -\text{sen}\theta & \cos\theta\cos\varphi \\ \text{sen}\theta\,\text{sen}\varphi & \cos\theta & \text{sen}\theta\cos\varphi \\ \cos\varphi & 0 & -\text{sen}\varphi \end{vmatrix} =$$

$$= r^2 \operatorname{sen}\varphi \cdot \left(+ \operatorname{sen}\theta \begin{vmatrix} \operatorname{sen}\theta \operatorname{sen}\varphi & \operatorname{sen}\theta \cos\varphi \\ \cos\varphi & -\operatorname{sen}\varphi \end{vmatrix} + \cos\theta \begin{vmatrix} \cos\theta \operatorname{sen}\varphi & \cos\theta \cos\varphi \\ \cos\varphi & -\operatorname{sen}\varphi \end{vmatrix} \right)$$

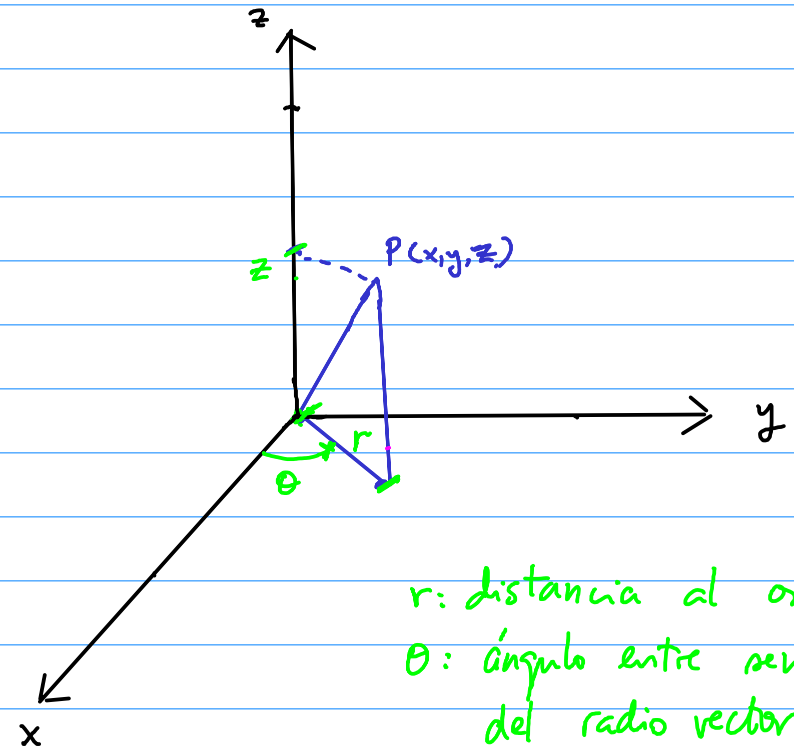
$$= r^2 \operatorname{sen}\varphi \Big[\operatorname{sen}\theta \left(-\operatorname{sen}^2\varphi \operatorname{sen}\theta - \cos^2\varphi \operatorname{sen}\theta \right) +$$

$$+ \cos\theta \left(-\operatorname{sen}^2\varphi \cos\theta - \cos^2\varphi \cos\theta \right) \Big] =$$

$$= r^2 \operatorname{sen}\varphi \left[-\operatorname{sen}^2\theta - \cos^2\theta \right] = -r^2 \operatorname{sen}\varphi$$

$$\left| \det J\phi(r,\theta,\varphi) \right| = r^2 \operatorname{sen}\varphi$$

COORDENADAS CILÍNDRICAS



r: distancia al origen de la proyección

θ: ángulo entre semieje x^+ y proy. del radio vector $\overrightarrow{OP}$

z: la misma que en c. cartesianas

$$\phi : (0,+\infty) \times (0,2\pi) \times \mathbb{R} \longrightarrow \mathbb{R}^3 \setminus A$$

$$(r, \theta, z) \longrightarrow (x,y,z) = (r\cos\theta, r\,\mathrm{sen}\,\theta, z)$$

$A = \{(x,y,z) \text{ tq } y=0,\ x \geq 0\}$

$$|J\phi(r,\theta,\varphi)| = \begin{vmatrix} \cos\theta & -r\,\mathrm{sen}\,\theta & 0 \\ \mathrm{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\,\mathrm{sen}^2\theta = r$$