Sistemas de Ecuaciones

3.1

a) $$\left.\begin{aligned} \alpha x + y + 2z &= 0 \\ x + 3y + z &= 0 \\ 3x + 10y + 4z &= 0 \end{aligned}\right\}$$

$$\overbrace{\left(\begin{array}{ccc|c} \alpha & 1 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 3 & 10 & 4 & 0 \end{array}\right)}^{A|b} \underset{A}{} \overset{F_{1,2}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ \alpha & 1 & 2 & 0 \\ 3 & 10 & 4 & 0 \end{array}\right) \xrightarrow[F_3 - 3F_1]{F_2 - \alpha F_1} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1-3\alpha & 2-\alpha & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\overset{F_{2,3}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1-3\alpha & 2-\alpha & 0 \end{array}\right) \xrightarrow{F_3 - (1-3\alpha)F_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1+2\alpha & 0 \end{array}\right)$$

Caso:

* Si $1+2\alpha \neq 0 \Leftrightarrow \boxed{\alpha \neq \frac{-1}{2}}$ entonces $rg\, A = rg\,(A|b) = 3$ y estamos ante un 'S.C.D' cuya solución es $x=y=z=0$

* Si $\alpha = \frac{-1}{2}$ entonces $rg\, A = rg\,(A|b) = 2$ y el sistema es S.C.I. Resolveremos con la ayuda de 1 parámetro $\lambda \in \mathbb{R}$:

$$z = \lambda$$
$$y = -z = -\lambda$$
$$x = -z - 3y = -\lambda + 3\lambda = 2\lambda$$

3.1. b

$$\begin{cases} 3x-y+2z=1 \\ x+4y+z=\beta \\ 2x-5y+\alpha z=-2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 1 & 4 & 1 & \beta \\ 2 & -5 & \alpha & -2 \end{array}\right) \overset{F_{1,2}}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 1 & \beta \\ 3 & -1 & 2 & 1 \\ 2 & -5 & \alpha & -2 \end{array}\right) \underset{F_3-2F_1}{\overset{F_2-3F_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 4 & 1 & \beta \\ 0 & -13 & -1 & 1-3\beta \\ 0 & -13 & \alpha-2 & -2-2\beta \end{array}\right)$$

$$\overset{F_3-F_2}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 1 & \beta \\ 0 & -13 & -1 & 1-3\beta \\ 0 & 0 & \alpha-1 & -3+\beta \end{array}\right)$$

Distinguimos los siguientes casos:

⊛ $\boxed{\alpha \neq 1}$ En este caso $rg\, A = rg\, (A|b) = 3$ y tenemos un S. C. D. Lo resolvemos:

$$z = \frac{\beta-3}{\alpha-1}$$

$$y = \frac{-1}{13}\left(1-3\beta+\frac{\beta-3}{\alpha-1}\right)$$

$$x = \beta - \frac{\beta-3}{\alpha-1} + \frac{4}{13}\left(1-3\beta+\frac{\beta-3}{\alpha-1}\right)$$

⊛ $\boxed{\alpha = 1}$ En este caso se distinguen dos subcasos:

Ⓐ $\beta = 3$: entonces $rg\, A = rg\,(A|b) = 2$ y tenemos S.C.I necesitando 1 parámetro λ para la resolución:

$$z = \lambda \; ; \quad y = \frac{-1}{13}\left(1 - 3\beta + \lambda\right) \; ;$$

$$x = \beta - \lambda + \frac{4}{13}\left(1 - 3\beta + \lambda\right)$$

Ⓑ $\beta \neq 3$: entonces $rg\, A = 2 \neq 3 = rg\,(A|b)$ y estamos ante un S.I.

3.1. e

$$(S)\begin{cases} 2\lambda x + \mu y + 2z = 1 \\ 2\lambda x + (2\mu - 1)\, y + 3z = 1 \\ 2\lambda x + \mu y + (\mu+3)\, z = 2\mu - 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2\lambda & \mu & 2 & 1 \\ 2\lambda & 2\mu-1 & 3 & 1 \\ 2\lambda & \mu & \mu+3 & 2\mu-1 \end{array}\right) \underset{F_3-F_1}{\overset{F_2-F_1}{\sim}} \left(\begin{array}{ccc|c} 2\lambda & \mu & 2 & 1 \\ 0 & \mu-1 & 1 & 0 \\ 0 & 0 & \mu+1 & 2\mu-2 \end{array}\right)$$

Se distinguen los siguientes casos:

1. $\lambda \neq 0,\ \mu \neq 1,\ \mu \neq -1.$ Entonces $rg\, A = rg\,(A|b) = 3$ y (S) es S.C.D. Las soluciones son:

$$z = \frac{2(\mu-1)}{\mu+1} \qquad x = \frac{1}{2\lambda}\left(1 - \frac{4(\mu-1)}{\mu+1} + \frac{2\mu}{\mu+1}\right)$$

$$y = \frac{-2}{\mu+1}$$

2. $\lambda = 0$ y el sistema queda como sigue:

$$\left(\begin{array}{ccc|c} 0 & \mu & 2 & 1 \\ 0 & \mu-1 & 1 & 0 \\ 0 & 0 & \mu+1 & 2(\mu-1) \end{array}\right)$$

En este caso $rg\, A = 2$ y calculamos $rg\,(A|b)$, para ello calculamos el determinante

$$\begin{vmatrix} \mu & 2 & 1 \\ \mu-1 & 1 & 0 \\ 0 & \mu+1 & 2(\mu-1) \end{vmatrix} = (\mu-1)(\mu+1) + 2(\mu-1)(\mu-2\mu+2)$$

$$= (\mu-1)\left[\mu+1-2\mu+4\right] = (\mu-1)(-\mu+5)$$

Entonces:

Ⓐ Si $\mu \neq 1$ y $\mu \neq 5$ $\quad rg(A|b)=3$ y (S) es S.I

Ⓑ Si $\mu = 1$ ó $\mu = 5$ $\quad rg(A|b)=2=rg(A)$ y (S) es S.C.I.

CASO $\lambda=0, \mu=1$

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \Rightarrow \begin{cases} z=0 \\ y=1 \\ x=a \in \mathbb{R} \end{cases}$$

CASO $\lambda=0 \quad \mu=5$

$$\left(\begin{array}{ccc|c} 0 & 5 & 2 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 6 & 8 \end{array}\right) \Rightarrow \begin{cases} z=4/3 \\ y=-1/3 \\ x=a \in \mathbb{R} \end{cases}$$

3. $\mu=1, \lambda\neq 0$

$$\left(\begin{array}{ccc|c} 2\lambda & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

Aquí $rg\,A = rg(A|b) = 2 \Rightarrow$

S.C.I.

$$\begin{cases} z=0 \\ y=a \in \mathbb{R} \\ x=(1-a)\dfrac{1}{2\lambda} \end{cases}$$

$\underline{4.\ \mu=-1,\ \lambda\neq 0}$

$$\left(\begin{array}{ccc|c} 2\lambda & -1 & 2 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right) \Rightarrow \quad S.I.$$

3.1, f)

$$(S)\quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -a & 3 & 4 \\ 3 & -3 & 4 & 7 \\ 5 & -a-b & 7 & 8+b \end{array}\right) \underset{\substack{F_4-5F_1}}{\overset{\substack{F_2-2F_1}}{\sim}} {\scriptstyle F_3-3F_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -a-2 & 1 & -2 \\ 0 & -6 & 1 & -2 \\ 0 & -a-b-5 & 2 & -7+b \end{array}\right)$$

$$\underset{F_4-2F_2}{\overset{F_3-F_2}{\sim}} \left(\begin{array}{ccc|c} x & y & z & \\ 1 & 1 & 1 & 3 \\ 0 & -(a+2) & 1 & -2 \\ 0 & a-4 & 0 & 0 \\ 0 & a-b-1 & 0 & -3+b \end{array}\right) \overset{C_{2,3}}{\sim}$$

$$\underbrace{\left(\begin{array}{ccc|c} x & z & y & \\ 1 & 1 & 1 & 3 \\ 0 & 1 & -(a+2) & -2 \\ 0 & 0 & a-4 & 0 \\ 0 & 0 & a-b-1 & b-3 \end{array}\right)}_{C^*}$$

with the first three columns marked C.

Distinguimos los siguientes casos:

① * $a \neq 4 \Rightarrow rg(C) = rg(A) = 3$

$$|C^*| = (a-4) \cdot \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & b-3 \end{vmatrix} = (b-3)(a-4)$$

** (1.1) $b=3,\ a \neq 4$ $\qquad rg(C^*) = 3 = rg(C) \Rightarrow$ S.C.D.

Ahora resolvemos: $\left.\begin{array}{l} y=0 \\ z=-2 \\ x = 3-y-z = 5 \end{array}\right\}$ $\boxed{\begin{array}{l} y=0 \\ z=-2 \\ x=5 \end{array}}$

** (1.2) $a \neq 4 , b \neq -3$ $\quad rg(C^*) = 4 \neq 3 = rg(C) \Rightarrow$ S.I.

② * $a = 4$

$$(S) \sim \begin{array}{c} \\ \\ \end{array}\left(\begin{array}{ccc|c} x & z & y & \\ 1 & 1 & 1 & 3 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -b+3 & b-3 \end{array}\right)$$

Distinguimos los siguientes casos:

** (2.1) $a=4, b\neq 3 \Rightarrow rg(A) = 3 = rg(A|b) \Rightarrow$ S.C.D y $\left\{\begin{array}{l} y=-1 \\ z=-8 \\ x=12 \end{array}\right.$.

** (2.2) $a=4, b=3 \Rightarrow rg(A) = 2 = rg(A|b) \Rightarrow$ S.C.I y se necesita 1 parámetro para la resolución (λ).

$y=\lambda$; $z = -2+6\lambda$; $x = 3-\lambda+2-6\lambda = 5-7\lambda$

3.1.g)

$$\left.\begin{aligned} \alpha x+\beta y+z&=1\\ x+\alpha\beta y+z&=\beta\\ x+\beta y+\alpha z&=1 \end{aligned}\right\}\ (S)$$

$$(A|b)=\left(\underbrace{\begin{array}{ccc} \alpha & \beta & 1\\ 1 & \alpha\beta & 1\\ 1 & \beta & \alpha \end{array}}_{A}\left|\begin{array}{c} 1\\ \beta\\ 1 \end{array}\right.\right) \underset{F_3-F_2}{\overset{F_1-\alpha F_2}{\sim}} \left(\begin{array}{ccc|c} 0 & \beta(1-\alpha^2) & 1-\alpha & 1-\alpha\beta\\ 1 & \alpha\beta & 1 & \beta\\ 0 & \beta(1-\alpha) & \alpha-1 & 1-\beta \end{array}\right)$$

$$\overset{F_{2,1}}{\sim}\left(\begin{array}{ccc|c} 1 & \alpha\beta & 1 & \beta\\ 0 & \beta(1-\alpha^2) & 1-\alpha & 1-\alpha\beta\\ 0 & \beta(1-\alpha) & \alpha-1 & 1-\beta \end{array}\right) \ \underset{\sim}{F_2-(1+\alpha)F_3}$$

$$\left(\begin{array}{ccc|c} 1 & \alpha\beta & 1 & \beta\\ 0 & 0 & -(\alpha-1)(\alpha+2) & (1-\alpha\beta)-(\alpha+1)(1-\beta)\\ 0 & \beta(1-\alpha) & \alpha-1 & 1-\beta \end{array}\right) \overset{F_{3,2}}{\sim}$$

$$\sim\left(\begin{array}{ccc|c} 1 & \alpha\beta & 1 & \beta\\ 0 & \beta(1-\alpha) & \alpha-1 & 1-\beta\\ 0 & 0 & -(\alpha-1)(\alpha+2) & (1-\alpha\beta)-(\alpha+1)(1-\beta) \end{array}\right)=(C|d)$$

$$|A|=|C|=-\beta(1-\alpha)(\alpha-1)(\alpha+2)=0 \Rightarrow \left\{\begin{array}{l} \alpha=1\\ \alpha=-2\\ \quad ó\\ \beta=0 \end{array}\right.$$

Así que distinguimos los siguientes casos:

$\boxed{C.1}$ $\alpha\neq 1$, $\alpha\neq -2$, $\beta\neq 0$ entonces $rg\, A = rg(A|b)=3$ y (S) es S.C.D.
Las soluciones son:

$$z=\frac{(1-\alpha\beta)-(\alpha+1)(1-\beta)}{(\alpha-1)(\alpha+2)};$$

$$y = \left[(1-\beta) - \frac{(1-\alpha\beta) - (\alpha+1)(1-\beta)}{\alpha+2} \right] \frac{1}{\beta(1-\alpha)} ;$$

$$x = \beta - \frac{(1-\alpha\beta) - (\alpha+1)(1-\beta)}{(\alpha-1)(\alpha+2)} - \frac{\alpha}{(1-\alpha)} \left((1-\beta) - \frac{(1-\alpha\beta) - (\alpha+1)(1-\beta)}{\alpha+2} \right)$$

$\boxed{C.2}$ $\alpha = 1$

$$(S) \sim \left(\begin{array}{ccc|c} 1 & \beta & 1 & \beta \\ 0 & 0 & 0 & 1-\beta \\ 0 & 0 & 0 & -1+\beta \end{array} \right)$$

Se distinguen dos casos:

(C.2.a) $\underline{\beta \neq 1 \text{ y } \alpha = 1}$ (S) es S.I

(C.2.b) $\underline{\beta = \alpha = 1}$ (S) es S.C.I

$$(S) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left\{ \begin{array}{l} z = a \in \mathbb{R} \\ y = b \in \mathbb{R} \\ x = 1 - a - b \end{array} \right.$$

$\boxed{C.3}$ $\underline{\alpha = -2}$

$$(S) \sim \left(\begin{array}{ccc|c} 1 & -2\beta & 1 & \beta \\ 0 & 3\beta & -3 & 1-\beta \\ 0 & 0 & 0 & 2+\beta \end{array} \right)$$

En este caso rg $A = 2$ y rg$(A|b) = 2$ si $\beta = -2$ y rg$(A|b) = 3$ si $\beta \neq -2$, luego:

$\underline{(C.3.a)}$ Si $\alpha = -2$ y $\beta \neq -2$ (S) es S.I.

(C.3.b) Si $\alpha=\beta=-2$ (S) es S.C.I y tenemos:

$$\begin{cases} z = a \in \mathbb{R} \\ y = \frac{1}{3}\left(3+3a+6\right) = 1+a+2 \\ x = -2 - a - 4 - 4a - 8 = -14-5a \end{cases}$$

$\boxed{C.4}$ $\underline{\beta=0}$

$$(S) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & \alpha-1 & 1 \\ 0 & 0 & -(\alpha-1)(\alpha-2) & -\alpha \end{array}\right)$$

En este caso tenemos que rg $A=2$ si $\alpha\neq 1$ y que rg $(A)=1$ cuando ocurre que $\alpha=1$. Por otro lado el rg$(A|b)$ coincide con el de la matriz que calculemos seguidamente el determinante:

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & \alpha-1 & 1 \\ 0 & -(\alpha-1)(\alpha-2) & -\alpha \end{vmatrix} = -\alpha(\alpha-1) + (\alpha-1)(\alpha-2) =$$

$$= (\alpha-1)(-\alpha+\alpha-2) = -2(\alpha-1) = 0$$

$\Leftrightarrow$ $\underline{\underline{\alpha=1}}$. Así que rg$(A|b)=3$ si $\alpha\neq 1$ y rg$(A|b)=2$ si $\alpha=1$. Distinguimos 2 casos:

C.4.a) $\alpha\neq 1$ y $\beta=0$. (S) es S.I ya que rg$(A)=1\neq 2=$rg$(A|b$

C.4.b) $\alpha=1$ y $\beta=0$ (S) es S.I porque rg$(A)=2\neq$ $3=$rg$(A|b)$.

3.5 a

$$\left.\begin{array}{l} kx + y + z + t = k \\ x + ky + z + t = k \\ x + y + kz + t = k \\ x + y + z + kt = k \end{array}\right\} (S)$$

$$\left(\begin{array}{cccc|c} k & 1 & 1 & 1 & k \\ 1 & k & 1 & 1 & k \\ 1 & 1 & k & 1 & k \\ 1 & 1 & 1 & k & k \end{array}\right) \underset{\substack{F_2 - F_3 \\ F_3 - F_4}}{\overset{F_1 - kF_4}{\sim}} \left(\begin{array}{cccc|c} 0 & 1-k & 1-k & (1+k)(1-k) & k(1-k) \\ 0 & k-1 & 1-k & 0 & 0 \\ 0 & 0 & k-1 & 1-k & 0 \\ 1 & 1 & 1 & k & k \end{array}\right)$$

$$\overset{F_2 + F_1}{\sim} \left(\begin{array}{cccc|c} 0 & 1-k & 1-k & (1+k)(1-k) & k(1-k) \\ 0 & 0 & 2(1-k) & (1+k)(1-k) & k(1-k) \\ 0 & 0 & k-1 & 1-k & 0 \\ 1 & 1 & 1 & k & k \end{array}\right) \overset{F_2 + 2F_3}{\sim}$$

$$\left(\begin{array}{cccc|c} 0 & 1-k & 1-k & (1+k)(1-k) & k(1-k) \\ 0 & 0 & 0 & (1-k)(3+k) & k(1-k) \\ 0 & 0 & k-1 & 1-k & 0 \\ 1 & 1 & 1 & k & k \end{array}\right) \underset{\substack{F_1 \to F_2 \\ F_2 \to F_4}}{\overset{F_4 \to F_1}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & k & k \\ 0 & 1-k & 1-k & (1+k)(1-k) & k(1-k) \\ 0 & 0 & k-1 & 1-k & 0 \\ 0 & 0 & 0 & (1-k)(3+k) & k(1-k) \end{array}\right)$$

Distinguimos ahora los siguientes casos:

1) $k \neq 1, -3 \Rightarrow rg\, A = rg(A|b) = 3$ y (S) es S.C.D

$$t = \frac{k}{3+k} = z$$

$$y + z + (1+k)t = k \Rightarrow y + (2+k)\frac{k}{3+k} = k$$

$$y = k\left(1 - \frac{2+k}{3+k}\right) = \frac{k}{3+k}$$

$$x+y+z+kt = k \Rightarrow x = k - (k+2)\cdot\frac{k}{3+k} = k\left(\frac{3+k-2-k}{3+k}\right)$$

$$= \frac{k}{3+k}$$

Luego: $\boxed{x = y = z = t = \frac{k}{3+k}.}$

2) $\underline{k=1}$

$$(S) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow rg A = rg(A|b) = 1 \Rightarrow \text{S.C.I.}$$

(3 parámetros)

Soluciones:

$x = a \in \mathbb{R}$

$y = b \in \mathbb{R}$ $\qquad t = 1 - a - b - c$

$z = c \in \mathbb{R}$

3) $\underline{k=-3}$

$$(S) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -3 & -3 \\ 0 & 4 & 4 & -8 & -12 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{array}\right) \Rightarrow (S) \text{ es S.I.}$$

3.5. b)

$$(S)\quad \left(\begin{array}{c|c} \underbrace{\begin{matrix} 2k+2 & 3 & k \\ 4k-1 & k+1 & 2k-1 \\ 5k-4 & k+1 & 3k-4 \end{matrix}}_{A} & \begin{matrix} k+4 \\ 2k+2 \\ k-1 \end{matrix} \end{array}\right)$$

$$|A| = \begin{vmatrix} 2k+2 & 3 & k \\ 4k-1 & k+1 & 2k-1 \\ 5k-4 & k+1 & 3k-4 \end{vmatrix} \overset{F_3-F_2}{=} \begin{vmatrix} 2k+2 & 3 & k \\ 4k-1 & k+1 & 2k-1 \\ k-3 & 0 & k-3 \end{vmatrix} =$$

$$= (k-3)\begin{vmatrix} 2k+2 & 3 & k \\ 4k-1 & k+1 & 2k-1 \\ 1 & 0 & 1 \end{vmatrix} \overset{C_3-C_1}{=} (k-3)\begin{vmatrix} 2k+2 & 3 & -k-2 \\ 4k-1 & k+1 & -2k \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= (k-3)\begin{vmatrix} 3 & -k-2 \\ k+1 & -2k \end{vmatrix} \overset{F_2-F_1}{=} (k-3)\begin{vmatrix} 3 & -k-2 \\ k-2 & -k+2 \end{vmatrix} =$$

$$= (k-3)(k-2)\begin{vmatrix} 3 & -k-2 \\ 1 & -1 \end{vmatrix} = (k-3)(k-2)(-3+k+2) =$$

$$= (k-1)(k-2)(k-3)$$

Distinguimos los siguientes casos:

⊛ $k \neq 1,\ k \neq 2,\ k \neq 3$ $\qquad rg(A) = rg(A|b) = 3 \Rightarrow$ S.C.D.

⊛ $k=1$

$$(S) \sim \left(\begin{array}{ccc|c} 4 & 3 & 1 & 5 \\ 3 & 2 & 1 & 4 \\ 1 & 2 & -1 & 0 \end{array}\right) \underset{F_2+F_3}{\overset{F_1+F_3}{\sim}} \left(\begin{array}{ccc|c} 5 & 5 & 0 & 5 \\ 4 & 4 & 0 & 4 \\ 1 & 2 & -1 & 0 \end{array}\right)$$

$$\underset{\frac{1}{4}F_2}{\overset{F_1-\frac{5}{4}F_2}{\sim}} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right)$$

$$\text{rg } A = 2 = \text{rg}(A|b) \Rightarrow \underline{\text{S.C.I.}}$$

⊛ $k=2$

$$(S) \sim \left(\begin{array}{ccc|c} 6 & 3 & 2 & 6 \\ 7 & 3 & 3 & 6 \\ 6 & 3 & 2 & 1 \end{array}\right) \underset{F_2-F_3}{\overset{F_1-F_3}{\sim}}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 5 \\ 1 & 0 & 1 & 5 \\ 6 & 3 & 2 & 1 \end{array}\right)$$

$$\text{rg } A = 2 \neq 3 = \text{rg}(A|b) \Rightarrow \underline{\text{S.I}}$$

$\underline{* \, k=3}$

$$(S) \sim \left(\begin{array}{ccc|c} 8 & 3 & 3 & 7 \\ 11 & 4 & 5 & 8 \\ 11 & 4 & 5 & 2 \end{array}\right) \overset{r_3 - r_2}{\sim} \left(\begin{array}{ccc|c} 8 & 3 & 3 & 7 \\ 11 & 4 & 5 & 8 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

$$rg(A) = 2 \neq 3 = rg(A|b) \Rightarrow \quad S.I.$$

3.6. d)

$$(S)\quad \left(\begin{array}{cccc|c} a & 2 & 3 & 1 & 6 \\ 1 & 3 & -1 & 2 & b \\ 3 & -a & 1 & 0 & 2 \\ 5 & 4 & 3 & 3 & 9 \end{array}\right) \underset{F_4-3F_1}{\overset{F_2-2F_1}{\sim}}$$

$$\left(\begin{array}{cccc|c} a & 2 & 3 & 1 & 6 \\ 1-2a & -1 & -7 & 0 & b-12 \\ 3 & -a & 1 & 0 & 2 \\ 5-3a & -2 & -6 & 0 & -9 \end{array}\right) \underset{F_4+6F_3}{\overset{F_2+7F_3}{\sim}}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & u \end{array} \\ \left(\begin{array}{cccc|c} a & 2 & 3 & 1 & 6 \\ 22-2a & -1-7a & 0 & 0 & b+2 \\ 3 & -a & 1 & 0 & 2 \\ 23-3a & -2-6a & 0 & 0 & 3 \end{array}\right) \end{array} \overset{F_{2,3}}{\sim}$$

$$\left(\begin{array}{cccc|c} a & 2 & 3 & 1 & 6 \\ 3 & -a & 1 & 0 & 2 \\ 22-2a & -1-7a & 0 & 0 & b+2 \\ 23-3a & -2-6a & 0 & 0 & 3 \end{array}\right)$$

$$\underbrace{\begin{matrix} 22-2a & -1-7a & 0 & 0 \\ 23-3a & -2-6a & 0 & 0 \end{matrix}}_{C}$$

$$|C| = \begin{vmatrix} 22-2a & -1-7a \\ 23-3a & -2-6a \end{vmatrix} \xrightarrow{F_2-F_1} \begin{vmatrix} 22-2a & -1-7a \\ 1-a & -1+a \end{vmatrix} =$$

20 x 0 -

$$= (22-2a)(-1+a) + (1+7a)(1-a) = (1-a)(2a-22+1+7a) =$$

$$(1-a)(9a-21)=0 \begin{cases} a=1 \\ a=\frac{21}{9}=\frac{7}{3} \end{cases}$$

Distinguimos 3 casos:

* $a \neq 1,\ a \neq 7/3$ $\quad$ $rg(A) = rg(A|b) = 4 \Rightarrow$ S.C.D.

* $a = 1$

$$(S) \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 6 \\ 3 & -1 & 1 & 0 & 2 \\ 20 & -8 & 0 & 0 & b+2 \\ 20 & -8 & 0 & 0 & 3 \end{array}\right) \xrightarrow{F_4-F_3} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 6 \\ 3 & -1 & 1 & 0 & 2 \\ 20 & -8 & 0 & 0 & b+2 \\ 0 & 0 & 0 & 0 & -b+1 \end{array}\right)$$

Se distinguen otra vez dos casos:

** $a=1,\ b=1$ $\quad$ $rg\,A = rg(A|b) = 3$ $\quad$ S.C.I

** $a=1,\ b\neq 1$ $\quad$ $rg\,A = 3 \neq 4 = rg(A|b)$ $\quad$ S.I

$* \ a = 7/3$

$$(S) \sim \left(\begin{array}{cccc|c} 7/3 & 2 & 3 & 1 & 6 \\ 3 & -7/3 & 1 & 0 & 2 \\ 52/3 & -52/3 & 0 & 0 & b+2 \\ 16 & -16 & 0 & 0 & 3 \end{array}\right) \overset{3 \cdot F_3}{\sim}$$

$$\left(\begin{array}{cccc|c} 7/3 & 2 & 3 & 1 & 6 \\ 3 & -7/3 & 1 & 0 & 2 \\ 52 & -52 & 0 & 0 & 3b+6 \\ 16 & -16 & 0 & 0 & 3 \end{array}\right) \overset{F_4 - \frac{16}{52} F_3}{\sim}$$

$$\left(\begin{array}{cccc|c} 7/3 & 2 & 3 & 1 & 6 \\ 3 & -7/3 & 1 & 0 & 2 \\ 52 & -52 & 0 & 0 & 3b+6 \\ 0 & 0 & 0 & 0 & \frac{60-48b}{52} \end{array}\right)$$

Se distinguen 2 casos:

$** \ a = 7/3, \ b = \frac{5}{4}$ $\quad \operatorname{rg} A = 3 = \operatorname{rg}(A|b) \Rightarrow$ S.C.I

$** \ a = 7/3, \ b \neq 5/4$ $\quad \operatorname{rg} A = 3 \neq 4 = \operatorname{rg}(A|b) \Rightarrow$ S.I

3.7

Lingote	Oro	Plata	Cobre
1	20	30	50
2	30	40	30
3	40	50	10

x_i es el peso que hay que tomar del lingote i. Entonces:

$$(S)\left\{\begin{array}{l} 0{,}2x_1+0{,}3x_2+0{,}4x_3=42 \\ 0{,}3x_1+0{,}4x_2+0{,}5x_3=57 \\ 0{,}5x_1+0{,}3x_2+0{,}1x_3=51 \end{array}\right.$$

Multiplicando las ecuaciones por 10 tenemos:

$$(S')\left(\begin{array}{ccc|c} 2 & 3 & 4 & 420 \\ 3 & 4 & 5 & 570 \\ 5 & 3 & 1 & 510 \end{array}\right) \underset{F_2-5F_3}{\overset{F_1-4F_3}{\sim}}$$

$$\left(\begin{array}{ccc|c} -18 & -9 & 0 & -1620 \\ -22 & -11 & 0 & -1980 \\ 5 & 3 & 1 & 510 \end{array}\right) \overset{11\cdot F_1-9\cdot F_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -22 & -11 & 0 & -1980 \\ 5 & 3 & 1 & 510 \end{array}\right) \overset{-\frac{1}{11}F_2}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 180 \\ 5 & 3 & 1 & 510 \end{array}\right)$$

(S) es S.C.I con lo cual hay infinitas soluciones. Sea $x_1 = a$ gramos, luego

$x_2 = 180 - 2\cdot a$ gramos $(40 < a < 90)$ y

$x_3 = 510 - 540 + 6a - 5a = a - 30$ gramos.

Como $0 < a - 30 < 100 \Rightarrow 30 < a$

Así que las soluciones son:

$x_1 = a$ gr.

$x_2 = 180 - 2a$ gr.

$x_3 = a - 30$ gr

con $40 < a < 90$

3.8 abc

$$\left.\begin{array}{l} a+b+c=6 \\ a=b+c \\ abc-cba=198 \end{array}\right\}$$

La última ecuación hay que reescribirla como ponemos:

$$\left.\begin{array}{l} a+b+c=6 \\ a=b+c \\ 100a+10b+c-(100c+10b+a)=198 \end{array}\right\} (S)$$

$$\Rightarrow \begin{cases} 2a=6 \Rightarrow a=3 \\ a=b+c \\ 100(a-c)+(c-a)= 99(a-c)=198 \Rightarrow a-c=2 \end{cases}$$

$$\Rightarrow \begin{cases} a=3 \\ a=b+c \\ a=2+c \end{cases} \Rightarrow \begin{cases} 3=b+c \\ 3=2+c \end{cases} \Rightarrow b-2=0 \Rightarrow \begin{cases} b=2 \\ c=1 \end{cases}$$

Así que el número es: **321**

3.9

120 horas / semana de torno
260 horas / semana de fresadora

Producto	Torno	Fresadore	Benef.
A	0.1 h	0.2 h	3
B	0.25 h	0.3 h	5
C	—	0.4 h	4

La producción necesaria sera x_a, x_b y x_c unidads de producto A, B y C respectivamente y entoncs:

$$\left.\begin{array}{l} 3x_a + 5x_b + 4x_c = 3800 \\ 0{,}1x_a + 0{,}25x_b = 120 \\ 0{,}2x_a + 0{,}3x_b + 0{,}4x_c = 260 \end{array}\right\} (S) \Rightarrow$$

$$\left(\begin{array}{ccc|c} 30 & 50 & 40 & 38000 \\ 10 & 25 & 0 & 12000 \\ 20 & 30 & 40 & 26000 \end{array}\right) \underset{F_3-2F_2}{\overset{F_1-3F_2}{\sim}} \left(\begin{array}{ccc|c} 0 & -25 & 40 & 2000 \\ 10 & 25 & 0 & 12000 \\ 0 & -20 & 40 & 2000 \end{array}\right)$$

$$\underset{\sim}{F_3-F_1} \left(\begin{array}{ccc|c} 0 & -25 & 40 & 2000 \\ 10 & 25 & 0 & 12000 \\ 0 & 5 & 0 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{l} 5x_b = 0 \Rightarrow x_b = 0 \\ 10x_a = 12000 \Rightarrow \\ \quad \Rightarrow x_a = 1200 \end{array}\right.$$

$$40x_c = 2000 \Rightarrow x_c = 50$$

Así que el resultado es producir:

- 0 unidads del producto B,
- 1200 " " " A,
- 50 " " " C.

3.10

$$\left.\begin{array}{l} kx+y+z=k \\ x+ky+z=0 \\ x+y+kz=k \end{array}\right\} (S)$$

$$(A|b)=\left(\begin{array}{ccc|c} k & 1 & 1 & k \\ 1 & k & 1 & 0 \\ 1 & 1 & k & k \end{array}\right) \underset{F_3-F_2}{\overset{F_1-kF_2}{\sim}} \left(\begin{array}{ccc|c} 0 & 1-k^2 & 1-k & k-k^2 \\ 1 & k & 1 & 0 \\ 0 & 1-k & k-1 & 0 \end{array}\right) \underset{F_{2,3}}{\overset{F_{1,2}}{\sim}}$$

$$\left(\begin{array}{ccc|c} 1 & k & 1 & 0 \\ 0 & 1-k & k-1 & 0 \\ 0 & (1+k)(1-k) & 1-k & k(1-k) \end{array}\right) \overset{F_3-(1+k)F_2}{\sim} \left(\begin{array}{ccc|c} 1 & k & 1 & 0 \\ 0 & 1-k & k-1 & 0 \\ 0 & 0 & 2-k-k^2 & k(1-k) \end{array}\right)$$

$$\underbrace{\hphantom{\begin{array}{ccc} 1 & k & 1 \end{array}}}_{C}$$

$$|A|=|C|=(1-k)(2-k-k^2)=0 \Rightarrow \begin{cases} 1-k=0 \iff k=1 \\ 2-k-k^2=0 \iff k=1 \end{cases}$$

Resolvemos $2-k-k^2=0$

k	0	1	2	3	4
$2-k-k^2$	2	0	1	0	3
¿k solución?	NO	SÍ	NO	SÍ	NO

$2-k-k^2=-(k-1)(k-3)$

Así que distinguimos 3 casos:

C1 $k \neq 1,3$, en este caso $rg\, A = rg\,(A|b)=3$ y (S) es S.C.D.

$$\begin{cases} (2-k-k^2)z = (1-k)(k-3)z = k(1-k) \Rightarrow z = k(k-3)^{-1} \\ (1-k)y+(k-1)z=0 \Rightarrow -y+z=0 \Rightarrow z=y \\ x=-z-ky=-k(k-3)^{-1}-k^2(k-3)^{-1} \end{cases}$$

Finalmente:

$$\begin{cases} z = y = k(k-3)^{-1} \\ x = -(k-3)^{-1} k(1+k) \end{cases}$$

C.2 $k=1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(S) es S.C.I. y tiene por soluciones:

$$\begin{cases} x = a \in \mathbb{Z}_5 \\ y = b \in \mathbb{Z}_5 \\ z = -a-b \end{cases}$$

¡¡ 25 soluciones !!

C.3. $k=3$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right) \Rightarrow$$ (S) es S.I