Detecting Chaos in a Duopoly Model Via Symbolic Dynamics

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Abstract

This paper considers a Cournot duopoly model assuming isoelastic demand and smooth cost functions with built-in capacity limits. When the firms cannot obtain positive profits they are assumed to choose small "stand-by" outputs rather than closing down, in order to avoid substantial fitting up costs when market conditions turn out more favorable. It is shown that the model provides chaotic behavior. In particular, the system has positive topological entropy and hence the map is chaotic in the Li–Yorke sense. Moreover, chaos is not only topological but also physically observable.

1 Introduction

This paper considers Cournot [2] duopoly dynamics for a model with isoelastic demand and cost functions (total and marginal) that asymptotically go to infinity as the capacity limits are approached. Isoelastic demand was suggested by one of the present authors [6] in 1991, and combined with constant marginal costs, it was shown to result in a period doubling cascade to chaos.

In some later publications, constant marginal costs (see [7], [8]) were considered unrealistic as they implied firms that were potentially infinite sized. In accordance with an argument by Edgeworth who in the late 19th Century insisted on the necessity of introducing capacity limits, a smooth model with asymptotic capacity limits has been proposed. It could be derived from a CES production function, so like the isoelastic demand function, which resulted from any Cobb-Douglas utility function, it was firmly based on generally accepted principles of economic modelling.

Another important point of the model we are going to propose is that we let firms to choose small "stand-by" outputs rather than closing down when they can not get positive profits. This fact is quite common in real life where firms tend not to close down when the

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market is unfavorable, but produce a small output in order to avoid substantial fitting up costs when market conditions turn out more favorable.

The rest of the paper is structures as follows. In Section 2 we give the definitions, notation and explain the model. Section 3 is devoted to introduce the necessary mathematical background to understand the mathematical analysis of the model, done in Section 4. Finally in Section 5 we give the final conclusions.

2 The Model

In this section we introduce the notation and basic definitions. We also construct the Cournot model and give the reaction functions.

With an isoelastic demand function, its inverse becomes

\[ p = \frac{R}{\sum_{i=2}^{q_i}}, \]

where \( p \) denotes market price, the \( q_i \) denote the supplies of the competitors, and \( R \) denotes the sum of the (constant) budget shares which all the consumers in aggregate spend on this particular commodity.

In order to make the formulas succinct we also define total market supply:

\[ Q = \sum_{i=1}^{i=2} q_i, \]  

and residual market supply

\[ Q_i = Q - q_i. \]

It is clear that in duopoly \( Q_1 = q_2 \) and \( Q_2 = q_1 \), whereas \( Q = q_1 + q_2 \).

Using (2) and (3) in (1), we obtain price as

\[ p = \frac{R}{Q_i + q_i}. \]

Total revenue for the \( i \)-th competitor then becomes

\[ R_i = \frac{Rq_i}{Q_i + q_i}, \]

and its derivative, the marginal revenue,

\[ \frac{dR_i}{dq_i} = \frac{RQ_i}{(Q_i + q_i)^2}. \]

As for cost, assume:

\[ C_i = c_i \frac{u_i^2}{u_i - q_i}, \]  

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where \( u_i \) denotes the capacity limit for the \( i \)-th competitor, and \( c_i \) denotes the initial marginal production cost. (See [7], [8] for how such cost functions are derived from CES production functions with capital fixed through some act of investment.)

The derivative is marginal cost,

\[
\frac{dC_i}{dq_i} = c_i \frac{u_i^2}{(u_i - q_i)^2}.
\]

Profit maximum is obtained when marginal revenue \( \frac{dR_i}{dq_i} \) equals marginal cost \( \frac{dC_i}{dq_i} \). From (5) and (7) we get that

\[
\frac{RQ_i}{(Q_i + q_i)^2} = \frac{c_i u_i^2}{(u_i - q_i)^2}
\]

or, given all variables and constants are non-negative, and \( u_i > q_i \),

\[
\frac{\sqrt{RQ_i}}{Q_i + q_i} = \sqrt{\frac{c_i u_i}{u_i - q_i}}
\]

From (9) we can easily solve for the optimal value of \( q_i \), the supply of the \( i \)-th competitor, as a function of \( Q_i \), the residual supply of the other competitor. This is Cournot’s reaction function:

\[
q'_i = f_i(Q_i) = u_i \frac{\sqrt{RQ_i} - \sqrt{c_i Q_i}}{\sqrt{RQ_i} + \sqrt{c_i u_i}}
\]

Negative profits make no sense either, otherwise the competitor will drop out. Profits are \( \Pi_i = R_i - C_i \), so, from (4) and (6),

\[
\Pi_i = \frac{Rq_i}{Q_i + q_i} - c_i \frac{u_i^2}{u_i - q_i}
\]

Substituting for optimal reaction from (10), maximum profit becomes

\[
\Pi_i^* = u_i \frac{R - c_i u_i - 2 \sqrt{c_i RQ_i}}{u_i + Q_i},
\]

which is nonnegative if \( R - c_i u_i - 2 \sqrt{c_i RQ_i} \geq 0 \). Given that \( c_i u_i < R \) (which must hold when fixed costs are less than maximum obtainable revenue), this is equivalent to:

\[
Q_i \leq \frac{(R - c_i u_i)^2}{4Rc_i}
\]

Hence, we reformulate the map (10) as:

\[
q'_i = F_i(Q_i) = \begin{cases} 
  u_i \frac{\sqrt{RQ_i} - \sqrt{c_i Q_i}}{\sqrt{c_i u_i} + \sqrt{RQ_i}}, & Q_i \leq \frac{(R - c_i u_i)^2}{4Rc_i} \\
  0, & Q_i > \frac{(R - c_i u_i)^2}{4Rc_i}
\end{cases}
\]
2.1 Properties of the Reaction Functions

The reaction functions \( q_i^0 = f_i(Q_i) \) as stated in (10) contain one positive square root term of \( Q_i \) and one negative linear term of \( Q_i \) in the numerator, which results in a unimodal shape, starting in the origin, having a unique maximum, and then dropping down to the axis. At the point \( Q_i = \frac{(R-c_i u_i)^2}{4c_i} \), there is a discontinuity, where the function drops down to the axis, as stated in (13). According to (13) the reaction curve drops down to zero. The reaction functions intersect at the origin, but this intersection of the reaction functions is unstable, which we see by calculating the derivatives

\[
\lim_{Q_i \to 0} \frac{df_i}{dQ_i}(Q_i) = \infty.
\]

Despite this the computer in a numerical experiment might stick to the origin as it is interpreted as an exact number. In reality it would be constantly perturbed and so diverge fast from this spurious equilibrium with infinite derivatives. Further, there are factual reason to assume that a firm that cannot obtain profits from staying in business will not close down completely, but rather choose a small "stand by" output. The particular reason for this is that even if nothing is produced, the putting up of a plant to full production again once the market is more favorable incurs considerable fitting up costs whenever it has been completely idle. (Even if the fixed capital costs cannot be avoided, these fixed fitting up costs can.)

Therefore we assume in the sequel that rather than putting \( q_i^0 = 0 \), the firms choose \( q_i^0 \in (0, \varepsilon_i) \), where \( \varepsilon_i \) is some small positive number. So, we will consider a duopoly game with the following set valued reaction functions

\[
q_1' = F_1(q_2) = \begin{cases} \frac{u_1 \sqrt{Rq_2} - \sqrt{c_1 q_2}}{\sqrt{c_1 u_1 + \sqrt{Rq_2}}} & \text{if } q_2 \leq \frac{(R-c_1 u_1)^2}{4Rc_1}, \\ F_1(q_2) \in (0, \varepsilon_1) & \text{if } q_2 > \frac{(R-c_1 u_1)^2}{4Rc_1}, \end{cases}
\]

\[
q_2' = F_2(q_1) = \begin{cases} \frac{u_2 \sqrt{Rq_1} - \sqrt{c_2 q_1}}{\sqrt{c_2 u_2 + \sqrt{Rq_1}}} & \text{if } q_1 \leq \frac{(R-c_2 u_2)^2}{4Rc_2}, \\ F_2(q_1) \in (0, \varepsilon_2) & \text{if } q_1 > \frac{(R-c_2 u_2)^2}{4Rc_2}. \end{cases}
\]

where

\[
\varepsilon_i < \max \left\{ F_i(Q_i) : Q_i \in \left[ 0, \frac{(R-c_i u_i)^2}{4Rc_i} \right] \right\}, \ i = 1, 2.
\]

Then, the functions are multivalued when \( Q_i > \frac{(R-c_i u_i)^2}{4Rc_i} \), because it can take any value in the interval \((0, \varepsilon_i)\).

3 Mathematical background

Proving in a mathematical rigorous manner that a system is chaotic needs some definitions on chaos. So, this section is devoted to introduce some basic notions.

A continuous map \( f : X \to X \) from a metric space \((X, d)\) into itself is said to be chaotic in the sense of Li–Yorke or simply \( LY \)-chaotic [4] if there is an uncountable set \( S \subseteq X \) such that for any \( x, y \in S, x \neq y \),

\[
\lim \inf_{n \to \infty} d(f^n(x), f^n(y)) = 0
\]
and
\[
\limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0.
\]
The set \( S \) is called a \textit{scrambled set} and \((x, y)\) is a Li–Yorke pair. LY–chaos is probably one of the most famous and accepted notions of chaos, closely related to topological entropy (see eg. [11, Chapter 7]). The definition is applied below to compact metric spaces.

Fix \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). A subset \( E \) is said to be \((n, \varepsilon)\)--separated if for any \( x, y \in E, x \neq y \), there is \( k \in \{0, 1, \ldots, n-1\} \) such that \( d(f^k(x), f^k(y)) > \varepsilon \). Denote by \( s_n(\varepsilon, f) \) the cardinality of an \((n, \varepsilon)\)--separated subset of maximal cardinality. Then the \textit{topological entropy} of \( f \) is defined to be
\[
h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, f).
\]
The following properties of topological entropy will be useful as can be found in [11, Chapter 7],
\[
h(f^k) = kh(f), \tag{15}
\]
where \( f^1 = f \) and \( f^k = f \circ f^{k-1} \). If a continuous map \( F : X \times Y \to X \times Y \) is given by \( F(x, y) = (f(x), g(x, y)) \) for any \((x, y) \in X \times Y\) (the map \( F \) is often known as skew–product map or triangular map), then
\[
h(f) \leq h(F). \tag{16}
\]

It is commonly accepted that positive topological entropy is an evidence of topological chaos. One motivation for this is the relationship of topological entropy with LY–chaos to the special case of continuous interval maps, that is, continuous maps \( f : [0,1] \to [0,1] \). Among other things, it has been proved that \( h(f) > 0 \) implies LY–chaoticity although the converse is false (see eg. [5] or [10]). This idea has been recently extended to continuous maps on compact metric spaces by proving that if the continuous map \( f : X \to X \) on a compact metric space \( X \) has positive topological entropy, then it is LY–chaotic [1]. Our aim will be to prove that our model has positive topological entropy and therefore is LY–chaotic.

4 The model: Mathematical analysis

In this section we will study the dynamic behavior of the duopoly system defined in (14). In order to do so we will assume that if \( Q_i > \frac{(R-c_i u_i)^2}{4Rc_i} \), \( i = 1, 2 \), then
\[
F_i(Q_i) \in \{\underline{\alpha}_i, \overline{\alpha}_i\}, \quad i = 1, 2, \tag{17}
\]
where \( 0 < \underline{\alpha}_i \leq \overline{\alpha}_i \leq \varepsilon_i, \ i = 1, 2 \). This means that at most two stand by outputs are allowed for each firm when \( Q_i > \frac{(R-c_i u_i)^2}{4Rc_i} \), \( i = 1, 2 \). Note that if we prove that with this assumption the system is chaotic then the system as defined in (14) with an interval of possible stand by outputs is also chaotic. This is due to the fact that any orbit of this (smaller) system can be a possible orbit of the system with an interval of possible stand by outputs.
We are going to associate a dynamical system to the model with assumption (17) and prove the existence of chaos. More precisely, under these hypothesis, we will prove that the duopoly game is chaotic in the sense of Li–Yorke.

To this end, we are going to rewrite the system as follows. For \( i = 1, 2 \), we consider the maps

\[
\overline{F}_i(Q_i) = \begin{cases} F_i(Q_i) & \text{if } Q_i \leq \frac{(R-c_i)u_i^2}{4Rc_i}, \\ \alpha_i & \text{if } Q_i > \frac{(R-c_i)u_i^2}{4Rc_i}, \end{cases}
\]

and

\[
\overline{F}'_i(Q_i) = \begin{cases} F_i(Q_i) & \text{if } Q_i \leq \frac{(R-c_i)u_i^2}{4Rc_i}, \\ \alpha_i & \text{if } Q_i > \frac{(R-c_i)u_i^2}{4Rc_i}, \end{cases}
\]

and for \( 1 \leq i \leq 4 \) define the maps \( G_i : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty) \), by

\[
G_1 = (\overline{F}_1, \overline{F}_2) \\
G_2 = (\overline{F}_1, F_2) \\
G_3 = (F_1, \overline{F}_2) \\
G_4 = (F_1, F_2).
\]

Note that \( F'_i(Q_i) = \overline{F}'_i(Q_i) = F_i(Q_i) \) if \( Q_i \leq \frac{(R-c_i)u_i^2}{4Rc_i}, \ i = 1, 2 \). Moreover, any orbit of any map \( G_i, 1 \leq i \leq 4 \), is finite because it eventually goes to a periodic point. To write the model in a suitable way, we need the following notation.

Let \( \Sigma_4 = \{1, 2, 3, 4\}^\mathbb{N} = \{(x_n) : x_n \in \{1, 2, 3, 4\}, \ n \geq 0\} \), be endowed by the metric

\[
d((x_n), (y_n)) = \frac{1}{4^i},
\]

where \( i \) is the first integer such that \( x_i \neq y_i \). The one side shift map \( \sigma_4 : \Sigma_4 \rightarrow \Sigma_4 \) defined by \( \sigma_4(x_n) = (x_{n+1}) \) is a continuous map. Now, we define the map

\[
H : \Sigma_4 \times [0, \infty) \times [0, \infty) \rightarrow \Sigma_4 \times [0, \infty) \times [0, \infty)
\]

by

\[
H((x_n), (q_1, q_2)) = (\sigma_4(x_n), G_{x_1}(q_1, q_2)) \text{ for all } ((x_n), (q_1, q_2)) \in \Sigma_4 \times [0, \infty) \times [0, \infty).
\]

Hence, if, for instance, \( (x_n) \) is the periodic sequence \((1, 2, 3, 4, 1, 2, 3, 4, \ldots)\), then, for any \( (q_1, q_2) \in [0, \infty) \times [0, \infty) \) the map \( H \) applies the maps \( G_i, 1 \leq i \leq 4 \), periodically following the order given by \( (x_n) \).

Denote by \( G^n_{x_1} = G_{x_1} \circ \ldots \circ G_{x_1}, n \geq 1 \) and let \( G^0_{x_1} \) be the identity map. Let \( \pi_1(x, y) = x \) and \( \pi_2(x, y) = y \) be the natural projections.

Now we are going to define a subset \( S_1 \subseteq \Sigma_4 \) as the subset formed by all the sequences \( (x_n) \) satisfying the following conditions:

(C1) There exists \( (q_1, q_2) \in [0, \infty) \times [0, \infty) \) such that if \( \pi_i(G^n_{x_1}(\pi_1(q_1, q_2))) < \frac{(R-c_i)u^2}{4Rc_i}, \ i = 1, 2, \) then \( x_n = x_{n+1}, n \geq 0 \).
(C2) If \( x_n \neq x_{n+1} \) and \( \pi_1 \left( G_{x_n}^n((q_1, q_2)) \right) < \frac{(R-c_1 u_i)^2}{4 R c_1} \) (respectively \( \pi_2 \left( G_{x_n}^n((q_1, q_2)) \right) < \frac{(R-c_1 u_i)^2}{4 R c_1} \)), then \( \pi_2 \circ G_{x_n} \neq \pi_2 \circ G_{x_{n+1}} \) (respectively \( \pi_1 \circ G_{x_n} \neq \pi_1 \circ G_{x_{n+1}} \)), for \( n > 0 \).

Consider the restriction
\[
H|_{S_1 \times [0, \infty)^2} : S_1 \times [0, \infty)^2 \to S_1 \times [0, \infty)^2,
\]
and denote again by \( H \) the restricted map. Note that if \( (x_n) \in S_1 \), then \( \sigma_4(x_n) \in S_1 \) and hence the new map \( H \) is well defined. In addition, it is easy to check that the set \( S_1 \) is defined such that for any initial condition \( (q_1, q_2) \), there is a sequence \( (x_n) \in S_1 \) which in any iteration step describes which map \( G_i, 1 \leq i \leq 4 \) has been chosen to compute its orbit.

Let
\[
\mathcal{O} = \text{Orb}_{G_1}(\overline{\alpha}_1, \overline{\alpha}_2) \cup \text{Orb}_{G_2}(\overline{\alpha}_1, \overline{\alpha}_2) \cup \text{Orb}_{G_3}(\overline{\alpha}_1, \overline{\alpha}_2) \cup \text{Orb}_{G_4}(\overline{\alpha}_1, \overline{\alpha}_2).
\]

Note that, since the maps \( F_i, i = 1, 2 \) are strictly increasing if \( 0 < Q_i \leq \frac{(R-c_1 u_i)^2}{4 R c_1} \), the set \( \mathcal{O} \) is finite. Let \( X_1, X_2 \subset [0, \infty) \), be the smallest sets such that \( \pi_1(\mathcal{O}) = X_1 \) and \( \pi_2(\mathcal{O}) = X_2 \). Therefore \( X_1 \) and \( X_2 \) are finite sets. It is immediate to check that if \( (q_1, q_2) \notin \{(x, y) : x = 0\} \cup \{(x, y) : y = 0\} \), then for any sequence \( (x_n) \in S_1 \), there is as \( n \in \mathbb{N} \) such that \( G_{x_n}^n(q_1, q_2) \in X_1 \times X_2 \). Hence, we introduce a new restriction \( \tilde{H} = H|_{S \times X_1 \times X_2} : S \times X_1 \times X_2 \to S \times X_1 \times X_2 \):

where the compact set \( S \subset S_1 \) contains all possible sequences given by points \( (q_1, q_2) \in X_1 \times X_2 \). The following example will help the reader to understand how the model works.

**Example 1.** Assume that \( c_i = u_i = 1 \), for \( i = 1, 2 \) and \( R = 2 \). Let \( \overline{\alpha}_1 = 0.05 \) and \( \overline{\alpha}_2 = 0.01 \), for \( i = 1, 2 \). Then \( \frac{(R-c_1 u_i)^2}{4 R c_1} = 0.125 \) for \( i = 1, 2 \) and \( F(0.05) > 0.125 \), \( F(0.01) < 0.125 \) and \( F^2(0.01) > 1.25 \). We analyze the symbols which can be associated to any initial conditions. First, note that
\[
\text{Orb}_{G_1}(0.05, 0.05) = \{(0.05, 0.05), (F(0.05), F(0.05))\},
\]
\[
\text{Orb}_{G_2}(0.01, 0.05) = \{(0.01, 0.05), (F(0.05), F(0.01)), (F^2(0.01), 0.05)\},
\]
\[
\text{Orb}_{G_3}(0.05, 0.01) = \{(0.05, 0.01), (F(0.01), F(0.05)), (0.05, F^2(0.01))\}
\]
and
\[
\text{Orb}_{G_4}(0.01, 0.01) = \{(0.01, 0.01), (F(0.01), F(0.01)), (F^2(0.01), F^2(0.01))\}.
\]

Now, we analyze the chaoticity of \( \sigma_4 : S \to S \) by taking in account the following fact. Since \( F^2(0.05) > 0.125 \) and \( F^3(0.01) > 0.125 \), we have that if we start with an initial condition from the set
\[
I = \{(0.05, 0.05), (0.01, 0.05), (0.05, 0.01), (0.01, 0.01)\},
\]

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then $x_0 \in \{1, 2, 3, 4\}$ and $\pi_i(G_{x_i}^6(q_1, q_2)) > 0.125$ for any $(q_1, q_2) \in I, i = 1, 2$. Then $x_6$ can be any element from $\{1, 2, 3, 4\}$. So, let $S_2$ be the set of sequences whose initial conditions are in $I$. Then it is easy to see that there are four possible symbols in any sequence $(x_n) \in S_2$ when $n = 6k, k \geq 1$. Hence $s_n(1/2, \sigma_4^6) = 4^n$ and therefore

$$h(\sigma_4^6) \geq \lim_{n \to \infty} \frac{1}{n} \log s_n(1/2, \sigma_4^6) = \log 4.$$  

Then, by (15) we have that $h(\sigma_4) \geq \frac{1}{6} \log 4$. The following pictures shows 75 values of $q_1$ and $q_2$ for the initial condition $(0.1, 0.1)$.

The following result shows that the pictures obtained in the above example have sense.

**Theorem 4.1.** The map

$$\tilde{H} : S \times X_1 \times X_2 \to S \times X_1 \times X_2$$

defined above has positive topological entropy and therefore is LY-chaotic.

**Proof.** For any $(q_1, q_2) \in X_1 \times X_2$ let $n_{(q_1, q_2)} \in \mathbb{N}$ be the first positive integer such that there are two sequences $(x_n), (y_n) \in S$ satisfying that $x_i = y_i, 1 \leq i < n_{(q_1, q_2)}$ and $x_{n_{(q_1, q_2)}} \neq y_{n_{(q_1, q_2)}}$. In other words, $n_{(q_1, q_2)}$ is the first integer where the map $G_i$ used for the iterations could be replaced by another one. Since the set $X_1 \times X_2$ is finite we can define

$$\tau = \max\{n_{(q_1, q_2)} : (q_1, q_2) \in X_1 \times X_2\}.$$  

Next we are going to show that the map $\tilde{H}$ has positive topological entropy and therefore by [1], $\tilde{H}$ is chaotic in the sense of Li–Yorke.
Fix \( n \in \mathbb{N} \) and \( \varepsilon < 1 \). We are going to estimate the cardinality of a maximal \((\tau_n, \varepsilon, \sigma_4)\)-separated set in \( S \). For \( (x_n) \in S \) we construct the block \((x_1, \ldots, x_{\tau n})\). Let \( E \subset S \) be a set containing one and only one sequence \((x_n)\) for any possible different block \((x_1, \ldots, x_{\tau n})\). Since \( \varepsilon < 1 \), the set \( E \) is \((n, \varepsilon, \sigma_4)\)-separated. By the definition of \( \tau \), for any \( 0 \leq i < n \), there is at least one iterate where the symbol may or may not change. Hence the number of different blocks is at least \( 2^{n-1} \) and therefore

\[
s_{\tau n}(\varepsilon, \sigma_4) \geq 2^{n-1}.
\]

Then

\[
h(\sigma_4) \geq \lim_{n \to \infty} \frac{1}{\tau n} \log s_{\tau n}(\varepsilon, \sigma_4) = \lim_{n \to \infty} \frac{n - 1}{\tau n} \log 2 = \frac{1}{\tau} \log 2.
\]

Now, by (16) we have that \( h(\tilde{H}) \geq h(\sigma_4) > 0 \), and therefore the proof concludes.\( \Box \)

Some remarks are now necessary in order to explain the model a little more.

**Remark 1.** We are interested in orbits of \((q_1, q_2) \in X_1 \times X_2\). These orbits are given by \( \pi(H^n((x_n), (q_1, q_2))) \), where \( \pi((x_n), (q_1, q_2)) = (q_1, q_2) \). Since the set is finite \( X_1 \times X_2 \), we have that the number of different possible orbits is given by the number of different elements in \( S \). Hence the dynamical complexity of orbits from the model is provided by the complexity of the shift map \( \sigma_4 : S \to S \).

**Remark 2.** LY–chaos as well as positive topological entropy are topological notions. Then, even in the well–known case of continuous interval maps (one dimensional dynamics), the existence of LY–chaos does not imply that chaos will be physically observable: there are examples of LY–chaotic interval maps for which almost all points (with respect to the Lebesgue measure) are attracted by a periodic orbit (see eg. [9]). In the presented model, any point in \((0, \infty) \times (0, \infty)\) eventually goes into \( X_1 \times X_2 \) under a finite number of iterations. Therefore, the chaos in this model is physically observable.

**Remark 3.** Chaoticity of the system is given by the possibility of choosing between two values \( \alpha_i < \bar{\alpha}_i, i = 1, 2 \). The following simple example shows that, not always we can choose, we get chaos. Let \( f : [0, 1] \to [0, 1] \) be the interval map given by

\[
f(x) = \begin{cases} f(x) \in [3/4, 1], & \text{if } 0 \leq x \leq 1/2, \\
x, & \text{if } 1/2 < x \leq 1. \end{cases}
\]

Clearly any orbit is eventually periodic and therefore the map cannot be chaotic.

**5 Conclusions**

The particular sense of the present model in terms of economics is that a duopoly firm, which during some period cannot make any positive profit, may not close down completely, but rather supply some small stand by quantity of output. This quantity may not be defined more precisely than belonging to some interval. The rationale for this is that fitting up certain capital equipment for production may involve substantial costs which accrue in a discontinuous manner the moment production is increased from zero to any positive output however small, after which costs increase in a smooth manner.
This means that we split the fixed costs in capital costs proper, which must be paid for in any case once the equipment is there, and fitting up costs which accrue when the equipment is actually taken in use. This distinction was once commonplace when the theory of production was based on cost functions. However, once these were replaced by production functions there was no practical way to include these costs in the models, and they were forgotten.

To judge the empirical relevance of fitting up costs, just consider those for a nuclear plant closed down for political (environmental) reasons, but taken in use again at some acute shortage of power. As these costs may be considered as most substantial, a firm involved in short run competition might not choose to closed down completely, but rather supply the market with some small output, how large is not important from the view of costs of the individual firm. However, if the system is sensitive, the exact quantity chosen may have considerable influence on the market dynamics. In the present setup, ultimately based on the very general case of iso-elastic demand (based on Cobb-Douglas type utility functions) and cost functions with capacity limits (based on CES production functions) the result is chaos, as proven in this paper.

References


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