# Analysis and numerical simulation of a nonlinear mathematical model for testing the manoeuvrability capabilities of a submarine 

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#### Abstract

The aim of this work is to provide a mathematical and numerical tool for the analysis of the manoeuvrability capabilities of a submarine. To this end, we consider a suitable optimal control problem with constraints in both state and control variables. The state law is composed of a highly coupled and nonlinear system of twelve ordinary differential equations. Control inputs appear in linear and quadratic form and physically are linked to rudders and propeller forces and moments. We consider a nonlinear Bolza type cost function which represents a commitment between reaching a final desired state and a minimal expense of control. In a first part, following recent ideas in [F. Periago and J. Tiago, A local existence result for an optimal control problem modeling the manoeuvring of an underwater vehicle, Nonlinear Analysis: Real World Applications (2009), doi:10.1016/j.nonrwa.2009.09.002], we prove a local existence result for the above mentioned optimal control problem. In a second part, we address the numerical resolution of the problem by using a descent method with projection and optimal step-size parameter. To illustrate the performance of the method proposed in this paper and to show its application in a real engineering problem we include three different numerical experiments for a standard manoeuvre.


Key words: Manoeuvrability control, manned submarine, optimal control problem, constraints on state and control, existence theory, numerical simulation.

## 1. Introduction - Problem Formulation

### 1.1. Motivation

During the preliminary state of design of an underwater vehicle, an important number of decisions must be taken to optimize the final behavior of the prototype. Among them, we focus on rudders and propeller characteristics which are the main properties governing the manoeuvrability capabilities of the vehicle. Optimality criteria for that behavior depend on the specific vehicle under consideration. In this work, we consider manned submarines with weight around 2500 tons and length between 60 and 100 meters. For this type of vehicle, the most important manoeuvre capabilities are: turning ability, fast course and depth changes, low noise generation when manoeuvring and moderate vertical movements. Concerning the last point, we notice that although our design requirements clearly define an ocean vehicle, however, some of the missions are within the littoral scenario and due to the size of the submarine, small vertical movements when turning is a very significative requirement.

On the other hand, up to the knowledge of the authors, this type of large submarines does not incorporate an autopilot which, however, is of a major importance in other types of unmanned underwater vehicles (AUVs and ROVs, for instance). The final design of an autopilot for an underwater vehicle is usually carried out by means of a closed-loop control system because this is the only way of correcting unforeseen errors in the mechanical systems of the vehicle and/or disturbances in the surroundings of it. Nevertheless, most of the times, a closed-loop control

[^0]system for a submarine requires, as a first step, the computation of a set of feasible and/or optimal trajectories, which typically are computed by using an open-loop control system, and which serve as reference trajectories to be followed by the feedback control law. Trajectory-tracking and path-following control strategies are based on this idea (see for instance [1]). The issue of designing robust autopilots for large manned submarines is therefore a pertinent one.

Finally, it is worth to mention that International Marine Organization (IMO) accepts numerical simulation results for ship models validation. The case of manned submarines still has not received IMO's attention, but obviously the same principle applies for this type of underwater vehicles.

The requirements described in the preceding three paragraphs justify the need of developing a solid mathematical tool of numerical simulation capable of giving a first answer to the above mentioned points. The present work is addressed towards that goal.

### 1.2. Vehicle Modeling

Concerning previous works on this subject, small unmanned underwater vehicles have received a lot of attention during the last decades (see for instance [2,3] and the references there in), but for large manned submarines (typically designed for military purposes) not so much has been published since the pioneering works [4,5] where the vehicle equations of motion were described for the first time. Let us now briefly describe those equations. Following [6], we consider the vector state

$$
\begin{equation*}
\mathbf{x}=(x, y, z, \phi, \theta, \psi, u, v, w, p, q, r) \in \mathbb{R}^{12} \tag{1}
\end{equation*}
$$

where $\eta=(x, y, z ; \phi, \theta, \psi)$ indicates the position and orientation of the submarine in a world fixed coordinate system, and $v=(u, v, w ; p, q, r)$ is the vector of linear and angular velocities measured in the body coordinate system. Using the usual SNAME notation $[4,5,7]$, the equations of motion are given by

$$
\left\{\begin{array}{l}
\eta^{\prime}(t)=J(\eta(t)) v(t)  \tag{2}\\
M v^{\prime}(t)+C(v(t)) v(t)+D(v(t)) v(t)+g(\eta(t))=\tau(\mathbf{u}(t))
\end{array}\right.
$$

Here $J$ is the transformation matrix for the kinematic equations, $M=M_{R B}+M_{A}$ includes rigid-body inertia matrix $M_{R B}$ plus the added inertia matrix $M_{A}, C(v)=C_{R B}(v)+C_{A}(v)$, with $C_{R B}$ the rigid-body Coriolis and Centripetal matrix and $C_{A}$ the matrix of hydrodynamic Coriolis and Centripetal terms due to added mass, $D(v)$ represents hydrodynamic damping due to vortex shedding and skin friction, $g(\eta)$ is the vector of restoring (i.e., gravitational and buoyant) forces and moments, and $\tau(\mathbf{u})$ is the vector of control inputs. For the specific vehicle considered in this work, the control vector is given by

$$
\begin{equation*}
\mathbf{u}(t)=\left(\delta_{b}(t), \delta_{s}(t), \delta_{r}(t), n(t)\right), \tag{3}
\end{equation*}
$$

where $\delta_{b}$ and $\delta_{s}$ represent, respectively, the angle of the bow and stern coupled planes, $\delta_{r}$ is deflection of rudder and $n$ is the propeller revolution. We refer to [7, 8, 9] for more details on the system (2), but for the sake of completeness we include in Appendix B the explicit expression of the dynamics equations of motion considered in this work. Kinematic equations can be found in $[6,7]$.

A first theoretical analysis of this model has been recently carried out in [6] and numerically in [8, 9]. In both cases, $n(t)$ is assumed to be constant, which is not completely realistic but it is just a starting point for a better understanding of the problem. The present work is therefore a continuation of those previous works. Since the main novelty corresponds to the introduction of $n(t)$ as a control variable, next we describe the way in which it appears in the equations of motion. Propeller forces $\left(T_{p}\right)$ and moments $\left(Q_{p}\right)$ may be modeled in different ways, but the most commonly used model is the one given by

$$
\left\{\begin{array}{l}
T_{p}=K_{T 0} D^{4} n^{2}(t)+K_{T J}\left(1-\omega_{f}\right) D^{3} u(t) n(t)+K_{T J^{2}}\left(1-\omega_{f}\right)^{2} D^{2} u^{2}(t)  \tag{4}\\
Q_{p}=K_{Q 0} D^{5} n^{2}(t)+K_{Q J}\left(1-\omega_{f}\right) D^{4} u(t) n(t)+K_{Q J^{2}}\left(1-\omega_{f}\right)^{2} D^{3} u^{2}(t) .
\end{array}\right.
$$

Here $u(t)$ is surge velocity (i.e., the seventh component of the vector state (1)), $D$ the propeller diameter, $\omega_{f}$ the wake fraction number, and the coefficients $K_{*}$ have been obtained experimentally by using a scale model. We refer to [7, Chapter 4] for more details on this propeller model and to [8] for specific values of the above parameters. From a mathematical point of view, the important fact is that $n(t)$ acts on the system in linear and quadratic form. We also
recall that the rest of components of the control vector $\mathbf{u}(t)$, i.e. $\delta_{b}(t), \delta_{s}(t), \delta_{r}(t)$, act in a pure quadratic form. In addition, $\mathbf{u}(t)$ must take its values in the compact and convex set

$$
\begin{equation*}
K=\left[-\frac{5 \pi}{36}, \frac{5 \pi}{36}\right] \times\left[-\frac{5 \pi}{36}, \frac{5 \pi}{36}\right] \times\left[-\frac{7 \pi}{36}, \frac{7 \pi}{36}\right] \times[0,2.5] \tag{5}
\end{equation*}
$$

As we will see later on, it is convenient to describe the action of the control variable on the system through a mapping

$$
\begin{align*}
\Phi: & K
\end{align*} \rightarrow \mathbb{R}^{8}, \quad\left(\mathbf{u}, \mathbf{u}^{2}\right) \equiv\left(\delta_{b}, \delta_{s}, \delta_{r}, n, \delta_{b}^{2}, \delta_{s}^{2}, \delta_{r}^{2}, n^{2}\right) .
$$

As for the state variable $\mathbf{x}(t)$, it is also realistic to consider the following constraints on the Euler angles

$$
-\frac{\pi}{4}<\phi<\frac{\pi}{4}, \quad-\frac{\pi}{6}<\theta<\frac{\pi}{6}, \quad 0<\psi<2 \pi .
$$

Due to the bounded nature of ocean, the first three components $(x, y, z)$ are also limited to some bounded rectangle. Finally, the physics of the problem also imposes a constraint on the rest of components (i.e, linear ( $u, v, w$ ) and angular ( $p, q, r$ ) velocities). To sum up, we can assume that the state variable satisfies the constraint

$$
\mathbf{x}(t) \in \Omega \subset \mathbb{R}^{12} \quad \text { for all } t \geq 0
$$

where $\Omega$ is a bounded and convex domain.
For theoretical reasons that we will describe in Section 2, it is also convenient to rewrite the state law (2) in the form

$$
\begin{equation*}
\mathbf{x}^{\prime}(t)=Q(\mathbf{x}(t)) \Phi(\mathbf{u}(t))+Q_{0}(\mathbf{x}(t)) \tag{7}
\end{equation*}
$$

where

$$
Q: \mathbb{R}^{12} \rightarrow \mathcal{M}^{12 \times 8} \quad \text { and } \quad Q_{0}: \mathbb{R}^{12} \rightarrow \mathbb{R}^{12}
$$

An explicit expression for $Q$ appears in (11). $Q_{0}$ is then easily obtained from the kinematic and dynamic equations of motion (see Appendix B).

### 1.3. Problem Formulation

In this subsection, we change a bit the notation for the components of the state and control variables to adapt these to the notation which is more commonly used in optimal control theory. Precisely, we shall denote

$$
\mathbf{x}(t)=\left(x_{1}(t), \cdots, x_{12}(t)\right) \in \mathbb{R}^{12} \quad \text { and } \quad \mathbf{u}(t)=\left(u_{1}(t), \cdots, u_{4}(t)\right) \in \mathbb{R}^{4}
$$

The manoeuvrability capabilities of an underwater vehicle described at the beginning may be formulated as the following optimal control problem: given a final time $t_{f}$ and a final target $\mathbf{x}^{t_{f}}=\left(x_{1}^{t_{f}}, \cdots, x_{12}^{t_{f}}\right) \in \Omega$, we look for a control $\mathbf{u}=\mathbf{u}(t) \in L^{\infty}\left(\left[0, t_{f}\right] ; K\right)$ such that at time $t_{f}$, the state variable $\mathbf{x}\left(t_{f}\right)$ reaches (or at least is close to) the final target $\mathbf{x}^{t_{f}}$. At the same time, we require a minimal expense of control during the whole manoeuvre.

In mathematical terms, we have the following nonlinear optimal control problem:

$$
\left(\mathrm{P}_{t_{f}}\right) \begin{cases}\begin{array}{l}
\text { Minimize in } \mathbf{u}: \\
\text { subject to }
\end{array} & I(\mathbf{u})=\sum_{j=1}^{12} \alpha_{j}\left(x_{j}\left(t_{f}\right)-x_{j}^{t_{f}}\right)^{2}+\sum_{j=1}^{4} \int_{0}^{t_{f}} \beta_{j} u_{j}^{2}(t) d t \\
& \begin{array}{l}
\mathbf{x}^{\prime}(t)=Q(\mathbf{x}(t)) \Phi(\mathbf{u}(t))+Q_{0}(\mathbf{x}(t)), \quad 0<t<t_{f} \\
\\
\\
\mathbf{x}(0)=\mathbf{x}^{0} \in \Omega \\
\\
\mathbf{x}(t) \in \Omega \quad \text { and } \quad \mathbf{u}(t) \in K, 0 \leq t \leq t_{f} .
\end{array}\end{cases}
$$

Here, $\alpha_{j} \geq 0$, and $\beta_{j}>0$ are penalty parameters.
Notice that this problem can also be formulated as an exact or approximate controllability problem. Two reasons persuade us of considering the problem as a controllability one: (i) up to the best knowledge of the authors, there are not satisfactory results in controllability theory (both in existence theory and numerical algorithms) for the case in which controls appear in nonlinear form, and (ii) at the practical point of view it is convenient to have at our disposal the penalty parameters $\alpha_{j}$ and $\beta_{j}$ to weight at convenience the two requirements of the problem (reaching a final target and the cost of using controls).

### 1.4. Organization of the rest of the paper

In Section 2, we prove a local in time existence result for problem $\left(\mathrm{P}_{t_{f}}\right)$. We shall apply a recent general existence result [10, Theorem 1.1] which can be adapted to our specific problem. More precisely, the existence result in [10, Theorem 1.1] does not include the case in which constraints on the state variable appear in the problem. We will indicate how this new ingredient may be overcome. We also would like to emphasize that the main difficulty to apply more classical results on existence theory for optimal control problems (see for instance [11]) arises in the size of our problem and in the hard non-linearities which are present (our state law is composed of a highly coupled and nonlinear system of twelve differential equations). As everyone who is familiar with optimization problems knows, the size of the problem is very important. Hence, for instance, being able to check the convexity condition on the orientor fields for our problem seems to be not an easy task. For the contrary, the sufficient condition in [10, Theorem 1.1] has an algebraic nature and, as showed in the next section, we are able to verify it.

Section 3 is devoted to the numerical resolution of $\left(\mathrm{P}_{t_{f}}\right)$. We use in a standard way a descent method with projection and optimal step-size parameter. This requires the computation at each iteration of the gradient of the cost function with respect to the control variable. To do that we apply the adjoint method which amounts to solve numerically the state law and a suitable backward ODE linear system for the adjoint state. Apart from the size of the problem, a mathematical difficulty appears when solving the adjoint equation: since our state law incorporates non-differentiable terms like absolute value and squared roots, the adjoint system has a discontinuous right-hand side. Fortunately, the transversality condition (see $[12,13]$ ) which is needed to ensure the existence and uniqueness of solution is satisfied. Numerical experiments include not only the propeller model described above, but another model which let the propeller turning in both directions. We recall that one of the main industrial applications of this work is to assist designers during the preliminary state of design. Therefore, it is quite nice to check the behavior of the vehicle when some of its components (in this case the propeller) change its properties. Unfortunately, the mathematical model for this new propeller does not satisfy the requirements of the existence theory for optimal control problems and hence we are unable to prove an existence result in this last case. Even so, numerical simulations are valuable for naval industry. We describe the model of the new propeller in Appendix A at the end of the paper.

## 2. Existence of solution for problem ( $\mathbf{P}_{t_{f}}$ )

The goal of this section is to prove the following result:

## Theorem 2.1. For $t_{f}>0$, small enough, problem $\left(P_{t_{f}}\right)$ has, at least, one solution.

Before starting with the proof of the theorem we need to introduce some previous notation. Consider the vector mapping

$$
\begin{aligned}
\Psi=\left(\psi_{j}\right): & \mathbb{R}^{8} \\
& \rightarrow \mathbb{R}^{4} \\
& \mathbf{m}
\end{aligned} \mapsto \Psi(\mathbf{m})=\left(m_{1}^{2}-m_{5}, m_{2}^{2}-m_{6}, m_{3}^{2}-m_{7}, m_{4}^{2}-m_{8}\right)
$$

where $\mathbf{m}=\left(m_{1}, \cdots, m_{8}\right)$. Then, it is clear that the set

$$
\Phi(K)=\{\Phi(\mathbf{u}): \mathbf{u} \in K\}
$$

where $\Phi$ is defined as in (6), is included in the zero level set of $\Psi$, i.e.,

$$
\Phi(K) \subset\left\{\mathbf{m} \in \mathbb{R}^{8}: \Psi(\mathbf{m})=0\right\}
$$

Following [6], set

$$
\begin{equation*}
\mathcal{N}(K, \Phi)=\left\{\mathbf{v} \in \mathbb{R}^{8}: \text { for each } \mathbf{u} \in K, \text { either } \nabla \Psi(\Phi(\mathbf{u})) \mathbf{v}=0 \quad \text { or } \quad \exists j \text { s.t. } \nabla \psi_{j}(\Phi(\mathbf{u})) \mathbf{v}>0\right\} \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{N}(\mathbf{c}, Q)=\left\{\mathbf{v} \in \mathbb{R}^{8}: Q \mathbf{v}=\mathbf{0} \quad \text { and } \quad \mathbf{c} \cdot \mathbf{v} \leq 0\right\}, \tag{9}
\end{equation*}
$$

where in our specific case the vector $\mathbf{c} \in \mathbb{R}^{8}$ and the matrix $Q \in \mathcal{M}^{12 \times 8}$ are given by

$$
\begin{equation*}
\mathbf{c}=\left(0,0,0,0, \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}\right) \tag{10}
\end{equation*}
$$

with $\beta_{j}>0$ the penalty parameters considered in the definition of problem $\left(\mathrm{P}_{t_{f}}\right)$, and

$$
Q=\left(\begin{array}{cccccccc}
Q_{71} x_{7}^{2} & Q_{72} x_{7}^{2} & 0 & Q_{6 \times 8} x_{7} & Q_{75} x_{7}^{2} & Q_{76} x_{7}^{2} & Q_{77} x_{7}^{2} & Q_{78}  \tag{11}\\
0 & 0 & Q_{83} x_{7}^{2} & Q_{84} x_{7} & 0 & 0 & 0 & Q_{88} \\
Q_{91} x_{7}^{2} & Q_{92} x_{7}^{2} & 0 & Q_{94} x_{7} & Q_{95} x_{7}^{2} & Q_{96} x_{7}^{2} & Q_{97} x_{7}^{2} & Q_{98} \\
0 & 0 & Q_{103} x_{7}^{2} & Q_{104} x_{7} & 0 & 0 & 0 & Q_{108} \\
Q_{111} x_{7}^{2} & Q_{112} x_{7}^{2} & 0 & Q_{114} x_{7} & Q_{115} x_{7}^{2} & Q_{116} x_{7}^{2} & Q_{117} x_{7}^{2} & Q_{118} \\
0 & 0 & Q_{123} x_{7}^{2} & Q_{124} x_{7} & 0 & 0 & 0 & Q_{128}
\end{array}\right),
$$

respectively. Notice that the matrix $Q=Q(\mathbf{x})$ depends on the state variable $\mathbf{x}$ whose components will be denoted from now on by $\mathbf{x}=\left(x_{1}, \cdots, x_{12}\right)$. More precisely, $Q$ only depends on the seventh component of $\mathbf{x}$, which physically corresponds to surge velocity. In the above matrix, $Q_{i j}$ refer to elements which are different from zero. Specific values for these coefficients have been obtained by using a scale model [8].

As is easy to imagine, the fundamental question to ensure the existence of solution for a general optimal control problem is the relation between state and control in both the state law and the cost function. For the problem under consideration, this relationship is established through the sets $\mathcal{N}(\mathbf{c}, Q)$ and $\mathcal{N}(K, \Phi)$ in such a way that if the following inclusion holds

$$
\begin{equation*}
\mathcal{N}(\mathbf{c}, Q(\mathbf{x})) \subset \mathcal{N}(K, \Phi) \tag{12}
\end{equation*}
$$

for each admissible state $\mathbf{x} \in \Omega$, then problem $\left(\mathrm{P}_{t_{f}}\right)$ has, at least, one solution. This sufficient condition was introduced for the first time in [10, Theorem 1.1]. See also [6, Theorem 2.1] for a version adapted to our specific setting. The proof of this result is based on several relaxations of the original problem which are obtained by using standard techniques in Young measures [15, 16] and classical results of existence theory for optimal control problems [11, 17]. Finally, condition (12) appears in an essential way to show that a relaxed solution of the original problem in terms of Young measures is, in fact, a Dirac type measure located at an admissible control $\mathbf{u}(t)$ for the original problem. This means that $\mathbf{u}(t)$ is a solution of the original problem. As indicated in the Introduction, the main differences of the existence theorem [10, Theorem 1.1] with respect to our specific problem is that in that case constraints on the state variable do not appear and, in addition, a global Lipschitz condition on the state law is assumed to hold. In our case, the Lipschitz condition is local and we have constraints on the state variable. Fortunately, the same techniques used in the proof of [10, Theorem 1.1] apply in our situation, but for the shake of completeness we next indicate the main differences.

As proved in [16, Th. 1.1] for more general cost functions, if $\left(\mathrm{P}_{t_{f}}\right)$ has no constraints on the state variable, then the problem

$$
\left(\mathrm{RP}_{t_{f}}\right) \text { Minimize in } \mu=\left\{\mu_{t}\right\}_{t \in\left(0, t_{f}\right)}: \quad \widetilde{I}(\mu)=\sum_{j=1}^{12} \alpha_{j}\left(x_{j}\left(t_{f}\right)-x_{j}^{t_{f}}\right)^{2}+\sum_{j=1}^{4} \int_{0}^{t_{f}} \beta_{j} \int_{K} \lambda_{j}^{2}(t) d \mu_{t}(\lambda) d t
$$

subject to

$$
\mathbf{x}^{\prime}(t)=\int_{K} Q(\mathbf{x}(t)) \Phi(\lambda) d \mu_{t}(\lambda)+Q_{0}(\mathbf{x}(t))
$$

and

$$
\operatorname{supp}\left(\mu_{t}\right) \subset K, \quad \mathbf{x}^{0} \in \Omega
$$

is a relaxation of $\left(\mathrm{P}_{t_{f}}\right)$. A crucial step in the proof is the fact that given a sequence of admissible controls $\mathbf{u}^{j} \in$ $L^{\infty}\left(\mathbb{R}_{+} ; K\right)$, its associated sequence of states $\mathbf{x}^{j}$ is also bounded. This enables one to consider a Young measure associated to the pair $\left\{\left(\mathbf{x}^{j}, \mathbf{u}^{j}\right)\right\}$. In our situation, we must also prove that the sequence $\mathbf{x}^{j}$ is in fact admissible for our problem, i.e., that it satisfies the constraints $\mathbf{x}^{j}(t) \in \Omega$ for all $t$. Indeed, following [16, p. 387] we obtain the following estimate

$$
\left\|\mathbf{x}^{j}(t)-\mathbf{x}^{0}\right\| \leq C_{1} t+L \int_{0}^{t}\left\|\mathbf{x}^{j}(s)-\mathbf{x}^{0}\right\| d s,
$$

where $C_{1}>0$ is a constant which depends on the bounds imposed by the set $K$, and $L>0$ is the Lipschitz constant which, in our case, is local but uniform with respect to $\mathbf{u}$. Then, using Gronwall's lemma we get

$$
\left\|\mathbf{x}^{j}(t)-\mathbf{x}^{0}\right\| \leq C_{1} t\left(1+L t e^{L t}\right)
$$

Hence, for $t$ small enough $\mathbf{x}^{j}(t)$ lies in $\Omega$ and therefore it is admissible for our problem.
The rest of the proof of the existence theorem [16, Th. 1.1] remains true with no additional changes. Hence, the same holds for the existence theorems in [10, Theorem 1.1] and [6, Theorem 2.1].

We are now in a position to prove our main existence result.
Proof of Theorem 2.1. We proceed in three steps:
Step 1: reduction to a Lagrange type optimal control problem. Notice that the existence result in $[6,10]$ that we will apply here is stated and proved for a Lagrange type cost function. However, in our problem ( $\mathrm{P}_{t_{f}}$ ) the cost function is of Bolza type. The transformation of a Bolza problem into Lagrange is standard [11, p. 11]. All we have to do is to introduce a new state variable, say $\widetilde{x}(t)$, which satisfies the initial value problem

$$
\begin{cases}\widetilde{x}(t)=0, & 0<t<t_{f} \\ \vec{x}(0)=\frac{1}{t_{f}} \sum_{j=1}^{12} \alpha_{j}\left(x_{j}\left(t_{f}\right)-x_{j}^{t_{f}}\right)^{2} . & \end{cases}
$$

Then, the new state variable $\overline{\mathbf{x}}(t)=(\widetilde{x}(t), \mathbf{x}(t)) \in \mathbb{R}^{13}$ is a solution of an ODE system which has the same structure as our original one (matrix $Q \in \mathcal{M}^{12 \times 8}$ is replaced by another one incorporating a first row with all its components equal to zero, and the new vector $Q_{0}$ incorporates a null first component). The original cost function is also replaced by the Lagrange type cost

$$
\int_{0}^{t_{f}}\left[\sum_{j=1}^{4} \beta_{j} u_{j}^{2}(t)+\widetilde{x}(t)\right] d t .
$$

What is important here is that this transformation does not affect the verification of condition (12) so that, in fact, we can work with our original cost function and state law.

Step 2: local well-posedness character of the state law. We must prove that for any initial condition $\mathbf{x}^{0} \in \Omega$ there exists a maximal time $t_{f}=t_{f}\left(\mathbf{x}^{0}\right)>0$ such that system

$$
(\operatorname{IVP})\left\{\begin{array}{l}
\mathbf{x}^{\prime}(t)=Q(\mathbf{x}(t)) \Phi(\mathbf{u}(t))+Q_{0}(\mathbf{x}(t)), \\
\mathbf{x}(0)=\mathbf{x}^{0} \in \Omega
\end{array}\right.
$$

is well-posed in the sense that there exists a unique solution defined in $\left[0, t_{f}\right]$ for every admissible control $\mathbf{u} \in$ $L^{\infty}\left(\mathbb{R}_{+} ; K\right)$. This analysis is completely analogous to the case considered in [6] so that we refer the reader to that reference. We just would like to emphasize that the constraint on the final time $t_{f}$ appears because $Q_{0}(\mathbf{x})$ includes functions which locally belong to the Sobolev space $W^{1, \infty}$ and therefore we can only ensure that the right-hand side in (IVP) is locally Lipschitz.

Step 3: verification of condition (12). This is the crucial step of the process. For a fixed $\mathbf{x} \in \Omega$, let us take $\mathbf{v}=\left(v_{1}, \cdots, v_{8}\right) \in \mathcal{N}(\mathbf{c}, Q(\mathbf{x}))$. Condition $Q(\mathbf{x}) \mathbf{v}=0$ transforms into the system of linear equations

$$
\left\{\begin{array}{l}
x_{2}^{2}\left(Q_{71} v_{1}+Q_{72} v_{2}+Q_{75} v_{5}+Q_{76} v_{6}+Q_{77} v_{7}\right)+x_{7} Q_{74} v_{4}+Q_{78} v_{8}=0  \tag{13}\\
x_{2}^{2} Q_{83} v_{3}+x_{7} Q_{84} v_{8}+Q_{88} v_{8}=0 \\
\left.x_{7}^{2} Q_{91} v_{1}+Q_{92} v_{4}+Q_{95} v_{5}+Q_{96} v_{6}+Q_{97} v_{7}\right)+x_{7} Q_{94} v_{4}+Q_{98} v_{8}=0 \\
x_{7}^{2} Q_{103} v_{3}+x_{7} Q_{104} v_{4}+Q_{108} v_{8}=0 \\
x_{7}^{2}\left(Q_{111} v_{1}+Q_{112} v_{2}+Q_{115}+Q_{116} v_{6}+Q_{117} v_{7}\right)+x_{7} Q_{114 v_{4}}+Q_{118} v_{8}=0 \\
x_{7}^{2} Q_{123} v_{3}+x_{7} Q_{124 v_{4}}+Q_{128} v_{8}=0
\end{array}\right.
$$

Hence,

$$
\begin{align*}
& v_{8}=\frac{-\left(x_{7}^{2} Q_{83} v_{3}+x_{7} Q_{84} v_{4}\right)}{Q_{88}}  \tag{14}\\
& v_{8}=\frac{-\left(x_{7}^{2} Q_{103} v_{3}+x_{7} Q_{104} v_{4}\right)}{Q_{108}}  \tag{15}\\
& v_{8}=\frac{-\left(x_{7}^{2} Q_{123} v_{3}+x_{7} Q_{124} v_{4}\right)}{Q_{128}} \tag{16}
\end{align*}
$$

Combining (14) with (15), and (15) with (16), yields

$$
\left\{\begin{array}{l}
x_{7}^{2}\left[\frac{Q_{83}}{Q_{88}}-\frac{Q_{103}}{Q_{108}}\right] v_{3}=x_{7}\left[\frac{Q_{104}}{Q_{108}}-\frac{Q_{84}}{Q_{88}}\right] v_{4} \\
x_{7}^{2}\left[\frac{Q_{103}}{Q_{108}}-\frac{Q_{123}}{Q_{128}}\right] v_{3}=x_{7}\left[\frac{Q_{124}}{Q_{128}}-\frac{Q_{104}}{Q_{108}}\right] v_{4}
\end{array}\right.
$$

Since for our specific coefficients

$$
\frac{Q_{124}}{Q_{128}}-\frac{Q_{104}}{Q_{108}}=\frac{Q_{104}}{Q_{108}}-\frac{Q_{84}}{Q_{88}}=0
$$

and

$$
\frac{Q_{103}}{Q_{108}}-\frac{Q_{123}}{Q_{128}} \neq \frac{Q_{83}}{Q_{88}}-\frac{Q_{103}}{Q_{108}}
$$

$v_{3}$ must be equal to zero and $v_{4}$ can take any value. Notice that an admissible state $\mathbf{x}(t) \in \Omega$ satisfies $x_{7}(t)>0$ for all $t \geq 0$. Moreover, since

$$
\frac{Q_{84}}{Q_{88}}=\frac{Q_{104}}{Q_{108}}=\frac{Q_{124}}{Q_{128}}=-0.066860778765595
$$

from (14), (15) and (16) we conclude that

$$
\begin{equation*}
v_{8}=-0.066860778765595 x_{7} v_{4} \tag{17}
\end{equation*}
$$

Reasoning as before, from (13) we obtain the following expressions for $v_{8}$

$$
\begin{align*}
& v_{8}=-x_{7}^{2}\left(\frac{Q_{71}}{Q_{78}} v_{1}+\frac{Q_{72}}{Q_{78}} v_{2}+\frac{Q_{75}}{Q_{78}} v_{5}+\frac{Q_{76}}{Q_{78}} v_{6}+\frac{Q_{77}}{Q_{78}} v_{7}\right)-x_{7} \frac{Q_{74}}{Q_{78}} v_{4}  \tag{18}\\
& v_{8}=-x_{7}^{2}\left(\frac{Q_{91}}{Q_{98}} v_{1}+\frac{Q_{92}}{Q_{98}} v_{2}+\frac{Q_{95}}{Q_{98}} v_{5}+\frac{Q_{96}}{Q_{98}} v_{6}+\frac{Q_{97}}{Q_{98}} v_{7}\right)-x_{7} \frac{Q_{94}}{Q_{98}} v_{4}  \tag{19}\\
& v_{8}=-x_{7}^{2}\left(\frac{Q_{111}}{Q_{118}} v_{1}+\frac{Q_{112}}{Q_{118}} v_{2}+\frac{Q_{115}}{Q_{118}} v_{5}+\frac{Q_{116}}{Q_{118}} v_{6}+\frac{Q_{117}}{Q_{118}} v_{7}\right)-x_{7} \frac{Q_{114}}{Q_{118}} v_{4} \tag{20}
\end{align*}
$$

Equating (18) and (19), and (19) with (20), gives

$$
\left\{\begin{array}{l}
\left(\frac{Q_{71}}{Q_{8}}-\frac{Q_{91}}{Q_{98}}\right) v_{1}+\left(\frac{Q_{72}}{Q_{7}}-\frac{Q_{92}}{Q_{98}}\right) v_{2}+\left(\frac{Q_{75}}{Q_{78}}-\frac{Q_{95}}{Q_{98}}\right) v_{5}+\left(\frac{Q_{76}}{Q_{78}}-\frac{Q_{96}}{Q_{98}}\right) v_{6}+\left(\frac{Q_{77}}{Q_{78}}-\frac{Q_{97}}{Q_{98}}\right) v_{7}=\frac{1}{Q_{1}}\left(\frac{Q_{94}}{Q_{98}}-\frac{Q_{74}}{Q_{78}}\right) v_{4}  \tag{21}\\
\left(\frac{Q_{91}}{Q_{98}}-\frac{Q_{111}}{Q_{118}}\right) v_{1}+\left(\frac{Q_{92}}{Q_{98}}-\frac{Q_{112}}{Q_{118}}\right) v_{2}+\left(\frac{Q_{05}}{Q_{98}}-\frac{Q_{115}}{Q_{118}}\right) v_{5}+\left(\frac{Q_{96}}{Q_{98}}-\frac{Q_{116}}{Q_{118}}\right) v_{6}+\left(\frac{Q_{97}}{Q_{98}}-\frac{Q_{117}}{Q_{118}}\right) v_{7}=\frac{1}{x_{7}}\left(\frac{Q_{114}}{Q_{118}}-\frac{Q_{94}}{Q_{98}}\right) v_{4} .
\end{array}\right.
$$

Since our coefficients satisfy

$$
\left\{\begin{array}{l}
\frac{Q_{95}}{Q_{98}}-\frac{Q_{115}}{Q_{11}}=\frac{Q_{75}}{Q_{75}}-\frac{Q_{95}}{Q_{98}}=0 \\
\frac{Q_{96}}{Q_{98}}-\frac{Q_{116}}{Q_{11}}=\frac{Q_{61}}{Q_{78}}-\frac{Q_{96}}{Q_{98}}=0 \\
\frac{Q_{97}}{Q_{98}}-\frac{Q_{117}}{Q_{118}}=\frac{Q_{77}}{Q_{78}}-\frac{Q_{97}}{Q_{98}}=0 \\
\frac{Q_{114}}{Q_{118}}-\frac{Q_{94}}{Q_{98}}=\frac{Q_{94}}{Q_{98}}-\frac{Q_{74}}{Q_{78}}=0
\end{array}\right.
$$

(21) transforms into

$$
\left\{\begin{array}{l}
\left(\frac{Q_{71}}{Q_{78}}-\frac{Q_{91}}{Q_{98}}\right) v_{1}+\left(\frac{Q_{72}}{Q_{8}}-\frac{Q_{92}}{Q_{98}}\right) v_{2}=0 \\
\left(\frac{Q_{91}}{Q_{98}}-\frac{Q_{111}}{Q_{118}}\right) v_{1}+\left(\frac{Q_{92}}{Q_{98}}-\frac{Q_{112}}{Q_{118}}\right) v_{2}=0 .
\end{array}\right.
$$

Finally, since the determinant of this system is different from zero, $v_{1}=v_{2}=0$.
As for condition $\mathbf{c} \cdot \mathbf{v} \leq 0$, taking (10) into account we obtain

$$
\begin{equation*}
\beta_{1} v_{5}+\beta_{2} v_{6}+\beta_{3} v_{7}+\beta_{4} v_{8} \leq 0 \tag{22}
\end{equation*}
$$

where $\beta_{j}>0$, for $j=1,2,3,4$. Substituting (17) into (22), we have

$$
\begin{equation*}
\beta_{1} v_{5}+\beta_{2} v_{6}+\beta_{3} v_{7}-0.066860778765595 x_{7} \beta_{4} v_{4} \leq 0 \tag{23}
\end{equation*}
$$

To sum up, if $Q(\mathbf{x}) \mathbf{v}=\mathbf{0}$ and $\mathbf{c} \cdot \mathbf{v} \leq 0$, then

$$
v_{1}=v_{2}=v_{3}=0,
$$

and (23) holds.
Let us now show that such a vector $\mathbf{v}$ belongs to the set $\mathcal{N}(K, \Phi)$ as given by (8). From the definition of the mapping $\Psi$,

$$
\nabla \Psi(\mathbf{m})=\left[\begin{array}{cccccccc}
2 m_{1} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 m_{2} & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 2 m_{3} & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 m_{4} & 0 & 0 & 0 & -1
\end{array}\right]
$$

Hence,

$$
\nabla \Psi(\mathbf{m}) \mathbf{v}=\left[\begin{array}{c}
-v_{5}  \tag{24}\\
-v_{6} \\
-v_{7} \\
2 m_{4} v_{4}+0.066860778765595 x_{7} v_{4}
\end{array}\right]
$$

Now take $\mathbf{v} \in \mathcal{N}(\mathbf{c}, Q(\mathbf{x}))$ and consider the three following cases:
$v_{4}=0$. From (23) we deduce that either $v_{5}=v_{6}=v_{7}=0$ or at least one of these three components is less than zero. In both cases (see (24)), we have $\mathbf{v} \in \mathcal{N}(K, \Phi)$.
$v_{4}>0$. Since $\mathbf{m} \in \Phi(K), m_{4} \geq 0$. As $x_{7}>0, \nabla \psi_{4}(\mathbf{m}) \mathbf{v}>0$ and therefore $\mathbf{v} \in \mathcal{N}(K, \Phi)$.
$v_{4}<0$. From (23) we deduce that at least one of the three components $v_{5}, v_{6}, v_{7}$ is negative. As before, $\mathbf{v} \in \mathcal{N}(K, \Phi)$.

## 3. Numerical resolution of problem $\left(\mathbf{P}_{t_{f}}\right)$

### 3.1. Algorithm of minimization

To simplify the notation, throughout this subsection we rewrite problem $\left(\mathrm{P}_{t_{f}}\right)$ in the compact form

$$
\begin{cases}\begin{array}{l}
\text { Minimize in } \mathbf{u}: \\
\text { subject to }
\end{array} & I(\mathbf{u})=G\left(\mathbf{x}\left(t_{f}\right)-\mathbf{x}^{t_{f}}\right)+\int_{0}^{t_{f}} F(\mathbf{u}(t)) d t \\
& \mathbf{x}^{\prime}(t)=\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad 0<t<t_{f} \\
& \mathbf{x}(0)=\mathbf{x}^{0} \in \Omega \\
& \mathbf{x}(t) \in \Omega \text { and } \mathbf{u}(t) \in K, 0 \leq t \leq t_{f} .\end{cases}
$$

As mentioned in the Introduction, a gradient descent method for solving numerically this problem is structured as:

1. Initialization of the control input: take an admissible $\mathbf{u}^{0}(t) \in K, 0 \leq t \leq t_{f}$.
2. For $k \geq 0$, iteration until convergence (e.g., $\left|I\left(\mathbf{u}^{k+1}\right)-I\left(\mathbf{u}^{k}\right)\right| \leq \varepsilon\left|I\left(\mathbf{u}^{0}\right)\right|$, with $0<\varepsilon \ll 1$ a prescribed tolerance) as follows:
2.1. once we have computed the gradient of the cost function $\nabla I\left(\mathbf{u}^{k}\right)$ and an optimal step-size parameter $\lambda^{k}>0$, we consider the vector

$$
\mathbf{v}^{k+1}=\mathbf{u}^{k}-\lambda^{k} \nabla I\left(\mathbf{u}^{k}\right)
$$

2.2. Since $\mathbf{v}^{k+1}$ may be not admissible, we consider its orthogonal projection over $K$, i.e,

$$
\mathbf{u}^{k+1}=\mathcal{P}_{K}\left(\mathbf{v}^{k+1}\right)
$$

where, for $K=\left[a_{1}, b_{1}\right] \times \cdots \times\left[a_{4}, b_{4}\right]$, the vector $\mathbf{u}=\mathcal{P}_{K}(\mathbf{v})$ has components

$$
u_{j}=\min \left(\max \left(a_{j}, v_{j}\right), b_{j}\right), \quad \mathbf{u}=\left(u_{j}\right), \mathbf{v}=\left(v_{j}\right), 1 \leq j \leq 4
$$

To compute, at each iteration $k$, the gradient $\nabla I\left(\mathbf{u}^{k}\right)$ we use the adjoint method as described next:

- Given the control $\mathbf{u}^{k}, k \geq 0$, solve the state equation

$$
\left\{\begin{array}{l}
\mathbf{x}^{\prime}(t)=\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}^{k}(t)\right), \quad 0 \leq t \leq t_{f}  \tag{25}\\
\mathbf{x}(0)=\mathbf{x}^{k}(0)
\end{array}\right.
$$

to obtain a new state $\mathbf{x}^{k+1}(t)$.

- With the pair $\left(\mathbf{u}^{k}, \mathbf{x}^{k+1}\right)$, solve the linear backward equation for the adjoint state $\mathbf{p}(t)$,

$$
\left\{\begin{array}{l}
\mathbf{p}^{\prime}(t)=-\left[\nabla_{\mathbf{x}} \mathbf{f}\left(\mathbf{x}^{k+1}(t), \mathbf{u}^{k}(t)\right)\right]^{T} \mathbf{p}(t), \quad 0 \leq t \leq t_{f}  \tag{26}\\
\mathbf{p}\left(t_{f}\right)=G^{\prime}\left(\mathbf{x}^{k+1}\left(t_{f}\right)-\mathbf{x}^{t_{f}}\right)
\end{array}\right.
$$

where $\nabla_{\mathbf{x}}$ is the gradient with respect to $\mathbf{x}$, and $A^{T}$ stands for the transpose of matrix $A$. We obtain $\mathbf{p}^{k+1}(t)$.

- Finally,

$$
\nabla I\left(\mathbf{u}^{k}\right)=\nabla_{\mathbf{u}} \mathbf{F}\left(\mathbf{x}^{k+1}(t), \mathbf{u}^{k}(t)\right)+\left[\nabla_{\mathbf{x}} \mathbf{f}\left(\mathbf{x}^{k+1}(t), \mathbf{u}^{k}(t)\right)\right]^{T} \mathbf{p}^{k+1}(t)
$$

As for the step-size parameter $\lambda^{k}$, it is chosen at each iteration in such a way that

$$
I\left(\mathbf{u}^{k}-\lambda^{k} \nabla I\left(\mathbf{u}^{k}\right)\right)=\min _{\lambda \in \mathbb{R}} I\left(\mathbf{u}^{k}-\lambda \nabla I\left(\mathbf{u}^{k}\right)\right) .
$$

At the numerical level, we approximate $I_{\lambda, k} \equiv I\left(\mathbf{u}^{k}-\lambda \nabla I\left(\mathbf{u}^{k}\right)\right)$ by the second-order polynomial

$$
m_{k}(\lambda)=a \lambda^{2}+b \lambda+c
$$

which satisfies

$$
m_{k}\left(\lambda_{1}=0.001\right)=I_{\lambda_{1}, k}, \quad m_{k}\left(\lambda_{2}=0.01\right)=I_{\lambda_{2}, k}, \quad m_{k}\left(\lambda_{3}=0.1\right)=I_{\lambda_{3}, k},
$$

so that

$$
\lambda^{k} \approx \arg \min _{\lambda \in\left[\lambda_{1}, \lambda_{3}\right]} m_{k}(\lambda)
$$

A delicate issue in the process just described corresponds to the numerical resolution of the adjoint system (26). The difficulty comes from the fact that the state law (25) includes non-differentiable terms like

$$
\left|x_{9}\right| \sqrt{x_{8}^{2}+x_{9}^{2}}
$$

so that $\nabla_{\mathbf{x}} \mathbf{f}$ has bounded discontinuities. As is well-known (see, for instance, [13, 14]), when the solution of a discontinuous ODE meets a discontinuity point, we may lost uniqueness of solutions, or the solution may be trapped by the discontinuity. A sufficient condition (the so-called transversality condition) to ensure that the solution traverses that discontinuity was introduced in [12]. See also [13, Section II.6]. Fortunately, in our case the transversality condition is satisfied. For the details we refer to [8] where the same phenomenon was observed.

Another point which deserves a comment concerns the fact that in the proposed algorithm of minimization we have not considered explicitly the constraints on the state variables. Our previous experimental experience with this type of submarines enables us to choose a final time $t_{f}$ for numerical simulation which leads to numerical results satisfying all the constraints of the problem. Nevertheless, our algorithm can be easily adapted to deal directly with state constraints. We refer to [18].

### 3.2. Numerical experiments

To illustrate the approach proposed in the present work, in this section we show three numerical experiments for a standard (in naval industry) manoeuvre. At each iteration of the gradient algorithm, the numerical resolutions of the state and adjoint state equations have been carried out by using the ODE45 MatLab function, which is a one-step solver based on an explicit Runge-Kutta method.

A typical manoeuvrability test for the type of submarine considered in this work is the so-called crash stop manoeuvre. It consists in decelerating the vehicle from an initial surge velocity, e.g. $u=7 \mathrm{~m} / \mathrm{s}$, to a final velocity close to, say $2 \mathrm{~m} / \mathrm{s}$. In addition, the submarine is forced to change its dept position from $z=400 \mathrm{~m}$ to (or close to) $z=50 \mathrm{~m}$, for instance. Next, we address the numerical simulation of this manoeuvre in three different cases: (1) in Experiment 1 we aim to test mathematically the proposed algorithm, i.e., we do not worry about the physical interest of the results but we pay all our attention on mathematical properties such a dependence with respect to initialization (in order to check the possible existence of several local and/or global minima) and exponential convergence of the algorithm. (2) In Experiment 2 and based on our experience, we show how to adapt a few parameters appearing in the formulation of problem $\left(\mathrm{P}_{t_{f}}\right)$ in order for the numerical simulation of the same crash stop manoeuvre to be relevant from a naval viewpoint. In both experiments we consider a fixed final time $t_{f}=250 s$ (for which all the constraints in state and control variables are satisfied) and show the results obtained by using the two propeller models described in this work. (3) In Experiment 3, we aim to compare the performance of the two propellers by considering the same crash stop manoeuvre but now looking for the minimum time $t_{f}$ for which at least one of the constraints of the problem is saturated. As well as our first experiment has a purely mathematical interest, the third one shows one of the possible applications of the proposed method during the preliminary state of design of the submarine. Indeed, the computation of such an optimal time for the manoeuvre is a typical example of a pre-contract navy requirement.

We would like to emphasize that since our mathematical model for the vehicle motion considers the most significant variables and forces that appear in a real situation, numerical results reflect the coupling effects which in fact are detected when navigating with a real submarine. Such coupling effects cannot be computed either with linear models or with nonlinear models with lower degrees of freedom. In this sense, up to knowledge of the authors, the present work is the first contribution addressing this issue.

In all the experiments that follow the stopping criterium of the algorithm is satisfied with $\varepsilon=10^{-6}$.

### 3.2.1. Experiment 1

In order to favor neither a particular component of the control variable nor the degree of accuracy in reaching the two final state components, we consider the following parameters

$$
\left\{\begin{array}{l}
\mathbf{x}(0)=(0,0,400,0,0,0,7,0,0,0,0,0) \\
t_{f}=250 \\
z\left(t_{f}\right) \equiv x_{3}(250)=50, \quad u\left(t_{f}\right) \equiv x_{7}(250)=2 \\
\alpha_{3}=\alpha_{7}=0.001, \quad \alpha_{j}=0, \quad j \neq 3,7 \\
\beta_{j}=0.001, \quad 1 \leq j \leq 4
\end{array}\right.
$$

The gradient algorithm is initiated with the following values:
$\left\{\right.$ (a) $\quad \mathbf{u}^{0}(t)=(-0.06,-0.84,0,2), 0 \leq t \leq 250$.
(b) $\mathbf{u}^{0}(t)=$ optimal controls for achieving a vertical movement, in $x_{3}$, from 400 m to 50 m .

We notice that the above values for $\mathbf{u}^{0}$ are physically compatible with the initial state $\mathbf{x}(0)$ of the vehicle.
Numerical results depend on initialization. Table 1 collects the minimum cost values for both initializations and for the two different propeller's models. Model 1 is the one defined in (4) and model 2 corresponds to (27). In addition, Figures 1-4 display the pictures for optimal controls with both initializations. The dash/dot line (-.-) represents the system with the propeller modeled by (4) and the solid line refers to the propeller model (27). We therefore conclude that our optimal control problem $\left(\mathrm{P}_{t_{f}}\right)$ presents several local minima.

The exponential convergence of the algorithm is illustrated in Figure 5 which shows the evolution of the cost function with respect to the number of iterations.

Since we have chosen the same weights in the cost function, numerical results for both initializations and propeller's models are not of physical interest. Indeed, since $\alpha_{3}=\alpha_{7}$, and there is a big difference in magnitude between

Table 1: Minimum cost for Experiment 1.

|  | model 1 | model 2 |
| :---: | :---: | :---: |
| Initialization (a) | 2.838 | 2.599 |
| Initialization (b) | 2.584 | 2.455 |



Figure 1: Rudder angles $\delta_{r}(t)$ for initialization (a) (Left) and (b) (Right).


Figure 2: Stern plane angles $\delta_{s}(t)$ for initialization (a) (Left) and (b) (Right).



Figure 3: Bow plane angle $\delta_{b}(t)$ for initialization (a) (Left) and (b) (Right).
the two state variables to be controlled ( $x_{3}$ must go from 400 to 50 , but $x_{7}$ just from 7 to 2 ) the algorithm focuses on minimizing $\left(x_{3}\left(t_{f}\right)-x_{3}^{t_{f}}\right)^{2}$. In practice, this difficulty is overcome by taking appropriate weights in the cost function. Next experiment is addressed towards that goal.


Figure 4: Propeller revolutions $n(t)$ for initialization (a) (Left) and (b) (Right).


Figure 5: Evolution of the cost function w.r.t. number of iterations for initialization (a) (Left) and (b) (Right).

### 3.2.2. Experiment 2

Consider now the following parameters

$$
\left\{\begin{array}{l}
\mathbf{x}(0)=(0,0,400,0,0,0,7,0,0,0,0,0) \\
t_{f}=250 \\
z\left(t_{f}\right) \equiv x_{3}(250)=50, \quad u\left(t_{f}\right) \equiv x_{7}(250)=2 \\
\alpha_{3}=0.0015, \alpha_{7}=10, \quad \alpha_{j}=0, j \neq 3,7 \\
\beta_{j}=0.001, \quad 1 \leq j \leq 4
\end{array}\right.
$$

and $\mathbf{u}^{0}(t)=(-0.06,-0.84,0,2), 0 \leq t \leq 250$.
As expected, this choice of the weights leads to physically admissible results. Figure 6 displays results for dept $x_{3}(t)$ and surge velocity $x_{7}(t)$. As before, the dash/dot line (-.-) represents the system with the propeller modeled by (4) and the solid line corresponds to the model given by (27). We also show the pictures for lateral movement $y$, roll angle $\phi$, pitch angle $\theta$, yaw angle $\psi$ as well as the controls rudder $\delta_{r}$, stern plane $\delta_{s}$, bow plane $\delta_{b}$ and propeller revolutions $n$ in Figure 7. Notice that the movement of rudder is very small. Nevertheless, the lateral movement of the vehicle is quite important. This result emphasizes the hight nonlinear and coupling effects existing in the vehicle model.

### 3.2.3. Experiment 3

In this experiment, we consider the same crash stop manoeuvre, but now we look for the minimum time for that manoeuvre. Of course, we must take into account our state and control constraints so that this minimum time depends on those. As we will see in the numerical results that follow, there is a minimal time for which one state variable saturates. In this specific example, it is pitch angle $\theta(t)$ which saturates at some time (see Figure 9 (c)). We consider the following parameters


Figure 6: Depth movement $z(t)($ Left $)$ and forward velocity $u(t)$ (Right) for Experiment 2.

$$
\left\{\begin{array}{l}
\mathbf{x}(0)=(0,0,400,0,0,0,7,0,0,0,0,0) \\
z\left(t_{f}\right) \equiv x_{3}\left(t_{f}\right)=50, \quad u\left(t_{f}\right) \equiv x_{7}\left(t_{f}\right)=2 \\
\alpha_{3}=0.005, \quad \alpha_{7}=45, \quad \alpha_{j}=0, j \neq 3,7 \\
\beta_{j}=0.001, \quad 1 \leq j \leq 4
\end{array}\right.
$$

Since we have two models for the propeller revolutions, there is a minimum time for each of them. For model 1 it is $210 s$, and for model 2 it is $145 s$. These results are consistent with the fact that in model 2 the propeller has the ability to turning in both senses and this is what helps to slow the forward speed down easily. The pictures for the behavior of some of the state variables and all of the control variables are displayed in Figure 9.

Concerning the computational cost for this experiment, in terms of computing time, model 1 requires 1393 seconds and model 2 takes 15 seconds in a PC with a core $2 d u o$ of 2.2 GHz and 4 GB of RAM.

Another interesting conclusion of the above three experiments is that in all cases optimal control variables are smooth which is a very nice feature from a naval point of view.

## A. A different propeller model

Propellers are usually asymmetric because its shape is optimized for producing the most significative thrust in forward direction. Let us denote by $V_{a}(t)=\left(1-\omega_{f}\right) u(t)$ the advanced speed at the propeller (here $\omega_{f}$ stands for the wake fraction number and $u(t)$ for the surge velocity of the vehicle) and by $n(t)$ the propeller revolutions (rps). For the case in which both $V_{a}$ and $n$ are non-negative, the thrust force $T_{p}$ and moment $Q_{p}$ produced by the propeller are modeled with the help of relations (4). This section describes a new model for a propeller that allows to turning in both senses, i.e., we consider the two cases: (1) $V_{a} \geq 0, n \geq 0$, and (2) $V_{a} \geq 0, n \leq 0$. Throughout this section we follow [19, Chapter 2].

The model is based on the angle of attack $\beta$ of the propeller blade at radius $0.7 R$ which is defined as

$$
\beta(t)=\arctan \left(\frac{V_{a}(t)}{0.7 \pi D n(t)}\right),
$$

where $D$ is the propeller diameter. Following [20], thrust force $T_{a}$ and moment $Q_{a}$ are given by

$$
\left\{\begin{array}{l}
T_{a}=\frac{1}{2} \rho C_{T}(\beta)\left(V_{a}^{2}+(0.7 \pi D)^{2}\right) \frac{\pi}{4} D^{2}  \tag{27}\\
Q_{a}=\frac{1}{2} \rho C_{Q}(\beta)\left(V_{a}^{2}+(0.7 \pi D)^{2}\right) \frac{\pi}{4} D^{3},
\end{array}\right.
$$

where $\rho$ is the water density. The non-dimensional thrust and torque coefficients $C_{T}(\beta)$ and $C_{Q}(\beta)$ are modeled by a $20^{\text {th }}$ order Fourier series in $\beta$ as follows:

$$
\begin{aligned}
& C_{T}(\beta)=\sum_{k=0}^{20}\left[A_{T}(k) \cos \beta k+B_{T}(k) \sin \beta k\right] \\
& C_{Q}(\beta)=\sum_{k=0}^{20}\left[A_{Q}(k) \cos \beta k+B_{Q}(k) \sin \beta k\right] .
\end{aligned}
$$



Figure 7: Results for Experiment 2.


Figure 8: Depth movement $z(t)($ Left $)$ and forward velocity $u(t)$ (Right) for Experiment 3.

Numerical values for the Fourier coefficients depend on the specific propeller and therefore may change if applied to other propellers. Since we do not have experimental values for these coefficients we have considered in our numerical simulations the values given in [19, Appendix A]. Those correspond to a propeller lightly different from our model. Figure 10 shows the pictures for the coefficients $C_{T}(\beta)$ and $C_{Q}(\beta)$ corresponding to our original model (in the range $0 \leq \beta \leq \pi / 2$ for which we have numerical values obtained by using a scale model) and to the model considered in [20] in the range $\pi \leq \beta \leq \pi$. We notice that for the type of submarine considered in this work, the range for $\beta$ is reduced to $0 \leq \beta \leq 1$ in the first case, and to $-1 \leq \beta \leq 1$ in the second one.

## B. Dynamic equations of motion

Following a standard notation in naval industry, we collect in this section the dynamic equation of motion that we have used in the paper. Numerical values for the non-dimensional hydrodynamic coefficients which appear in the equations below have been provided by the company Navantia S.A. and computed experimentally by using a scale model (see [8]).

AXIAL FORCE EQUATION:

$$
\begin{aligned}
& m\left[\dot{u}-v r+w q-x_{G}\left(q^{2}+r^{2}\right)+y_{G}(p q-\dot{r})+z_{G}(p r+\dot{q})\right] \\
= & \frac{\rho}{2} l^{4}\left[X_{q q}^{\prime} q^{2}+X_{r r}^{\prime} r^{2}+X_{r p}^{\prime} r p+X_{q \mid q q}^{\prime} q|q|\right]+\frac{\rho}{2} l^{3}\left[X_{\dot{u}}^{\prime} \dot{u}+X_{v r}^{\prime} v r+X_{w q}^{\prime} w q\right] \\
& +\frac{\rho}{2} l^{2}\left[X_{u u}^{\prime} u^{2}+X_{v v}^{\prime} v^{2}+X_{w w}^{\prime} w^{2}+X_{w|w|}^{\prime} w|w|\right] \\
& +\frac{\rho}{2} l^{2}\left[X_{\delta_{r} \delta_{r}}^{\prime} u^{2} \delta_{r}^{2}+X_{\delta_{s} \delta_{s}}^{\prime} u^{2} \delta_{s}^{2}+X_{\delta_{b} \delta_{b}}^{\prime} u^{2} \delta_{b}^{2}\right]-(W-B) \sin (\theta) \\
& +\rho T\left(1-t_{p}\right)
\end{aligned}
$$

where $T$ is thrust force produced by the propeller and is defined as in (4) for the first model, and as (27) for the second one

LATERAL FORCE EQUATION:

$$
\begin{aligned}
& m\left[\dot{v}-w p+u r-y_{G}\left(r^{2}+p^{2}\right)+z_{G}(q r-\dot{p})+x_{G}(q p+\dot{r})\right] \\
= & \frac{\rho}{2} l^{4}\left[Y_{r}^{\prime} \dot{r}+Y_{\dot{p}}^{\prime} \dot{p}+Y_{r|r| r}^{\prime} r|r|+Y_{p q}^{\prime} p q\right] \\
& +\frac{\rho}{2} l^{3}\left[Y_{r}^{\prime} u r+Y_{p}^{\prime} u p+Y_{\dot{v}}^{\prime} \dot{v}+Y_{w p}^{\prime} w p\right] \\
& +\frac{\rho}{2} l^{2}\left[Y_{*}^{\prime} u^{2}+Y_{v}^{\prime} u v+Y_{v| | \mid N}^{\prime} v\left|\left(v^{2}+w^{2}\right)^{\frac{1}{2}}\right|\right] \\
& +\frac{\rho}{2} l^{2}\left[Y_{\delta_{r}}^{\prime} u^{2} \delta_{r}+Y_{\delta_{r} \eta}^{\prime} u^{2} \delta_{r}\left(\eta-\frac{1}{C}\right) C\right] \\
& +\frac{\rho}{2} l^{2} Y_{v w N}^{\prime} v w+(W-B) \cos (\theta) \sin (\phi) \\
& 15
\end{aligned}
$$



Figure 9: Results for Experiment 3.


Figure 10: Coefficients for the propeller models (4) (Left) and (27) (Right).

## NORMAL FORCE EQUATION:

$$
\begin{aligned}
& m\left[\dot{w}-u q+v p-z_{G}\left(p^{2}+q^{2}\right)+x_{G}(r p-\dot{q})+y_{G}(r q+\dot{p})\right] \\
= & \frac{\rho}{2} l^{4}\left[Z_{\dot{q}}^{\prime} \dot{q}+Z_{q|q|}^{\prime} q|q|+Z_{r r}^{\prime} r^{2}\right]+\frac{\rho}{2} l^{3}\left[Z_{\dot{w}}^{\prime} \dot{w}+Z_{q}^{\prime} u q+Z_{v p}^{\prime} v p+Z_{v r}^{\prime} v r\right] \\
& +\frac{\rho}{2} l^{2}\left[Z_{*}^{\prime} u^{2}+Z_{w}^{\prime} u w+Z_{v v}^{\prime} v^{2}\right] \\
& +\frac{\rho}{2} l^{2}\left[Z_{|w|}^{\prime} u|w|+Z_{w w N}^{\prime}|w|\left(v^{2}+w^{2}\right)^{\frac{1}{2}}\right] \\
& +\frac{\rho}{2} l^{2}\left[Z_{\delta_{s}}^{\prime} u^{2} \delta_{s}+Z_{\delta_{b}}^{\prime} u^{2} \delta_{b}+Z_{\delta_{s} \eta}^{\prime} u^{2} \delta_{s}\left(\eta-\frac{1}{C}\right) C\right] \\
& +(W-B) \cos (\theta) \cos (\phi)
\end{aligned}
$$

## ROLLING MOMENT EQUATION:

$$
\begin{aligned}
& I_{x} \dot{p}+\left(I_{z}-I_{y}\right) q r-I_{z x} \dot{r}-I_{z x} p q+I_{y z} r^{2}-I_{y z} q^{2}+I_{x y} p r-I_{x y} \dot{q} \\
& m y_{G} \dot{w}-m y_{G} u q+m y_{G} v p-m z_{G} \dot{v}+m z_{G} w p-m z_{G} u r \\
= & \frac{\rho}{2} l^{5} K_{\dot{p}}^{\prime} \dot{p}+\frac{\rho}{2} l^{5} K_{r}^{\prime} \dot{r}+\frac{\rho}{2} l^{5} K_{q r}^{\prime} q r+\frac{\rho}{2} l^{5} K_{p|p|}^{\prime} p|p|+\frac{\rho}{2} l^{5} K_{r|r|}^{\prime} r|r| \\
& +\frac{\rho}{2} l^{4} K_{p}^{\prime} u p+\frac{\rho}{2} l^{4} K_{r}^{\prime} u r+\frac{\rho}{2} l^{4} K_{\dot{v}}^{\prime} \dot{v}+\frac{\rho}{2} l^{4} K_{w p}^{\prime} w p \\
& +\frac{\rho}{2} l^{3} K_{*}^{\prime} u^{2}+\frac{\rho}{2} l^{3} K_{v}^{\prime} u v+\frac{\rho}{2} l^{3} K_{v| | v \mid}^{\prime} v|v|+\frac{\rho}{2} l^{3} K_{\delta_{r}}^{\prime} u^{2} \delta_{r} \\
& +\left(Y_{G} W-Y_{B} B\right) \cos (\theta) \cos (\phi)-\left(Z_{G} W-Z_{B} B\right) \cos (\theta) \sin (\phi) \\
& -\rho Q
\end{aligned}
$$

where $Q$ is propeller moment (see (4) and (27)).

## PITCHING MOMENT EQUATION:

$$
\begin{aligned}
& I_{y} \dot{q}+\left(I_{x}-I_{z}\right) r p-(\dot{p}+q r) I_{x y}+\left(p^{2}-r^{2}\right) I_{z x}+(q p-\dot{r}) I_{y z} \\
& +m\left[z_{G}(\dot{u}-v r+w q)-x_{G}(\dot{w}-u q+v p)\right] \\
= & \frac{\rho}{2} l^{5}\left[M_{\dot{q}}^{\prime} \dot{q}+M_{r p}^{\prime} r p+M_{q|q|}^{\prime} q|q|+M_{r r}^{\prime} r^{2}\right]+\frac{\rho}{2} l^{4}\left[M_{\dot{w}}^{\prime} \dot{w}+M_{q}^{\prime} u q+M_{v r}^{\prime} v r\right] \\
& +\frac{\rho}{2} l^{3}\left[M_{*}^{\prime} u^{2}+M_{w}^{\prime} u w+M_{v v}^{\prime} v^{2}+M_{w|w| N}^{\prime} w\left|\left(v^{2}+w^{2}\right)^{\frac{1}{2}}\right|\right] \\
& +\frac{\rho}{2} l^{3}\left[M_{v w}^{\prime} v w+M_{|w|}^{\prime} u|w|+M_{w w}^{\prime}\left|w\left(v^{2}+w^{2}\right)^{\frac{1}{2}}\right|\right] \\
& +\frac{\rho}{2} l^{3}\left[M_{\delta_{s}}^{\prime} u^{2} \delta_{s}+M_{\delta_{b}}^{\prime} u^{2} \delta_{b}+M_{\delta_{s} \eta}^{\prime} u^{2} \delta_{s}\left(\eta-\frac{1}{C}\right) C\right] \\
& -\left(x_{G} W-x_{B} B\right) \cos (\theta) \cos (\phi)-\left(z_{G} W-z_{B} B\right) \sin (\theta)
\end{aligned}
$$

## YAWING MOMENT EQUATION:

$$
\begin{aligned}
& I_{z} \dot{r}+\left(I_{y}-I_{x}\right) p q-(\dot{q}+r p) I_{y z}+\left(q^{2}-p^{2}\right) I_{x y}+(r q-\dot{p}) I_{z x} \\
& +m\left[x_{G}(\dot{v}-w p+u r)-y_{G}(\dot{u}-v r+w q)\right] \\
= & \frac{\rho}{2} l^{5}\left[N_{\dot{r}}^{\prime} \dot{r}+N_{r|r|}^{\prime} r|r|+N_{\dot{p}}^{\prime} \dot{p}+N_{p q}^{\prime} p q\right]+\frac{\rho}{2} l^{4}\left[N_{p}^{\prime} u p+N_{r}^{\prime} u r+N_{\dot{v}}^{\prime} \dot{v}\right] \\
& +\frac{\rho}{2} l^{3}\left[N_{*}^{\prime} u^{2}+N_{v}^{\prime} u v+N_{v| | \mid N}^{\prime} v\left|\left(v^{2}+w^{2}\right)^{\frac{1}{2}}\right|\right] \\
& +\frac{\rho}{2} l^{[ }\left[N_{\delta_{r}}^{\prime} u^{2} \delta_{r}+N_{\delta_{r} \eta}^{\prime} u^{2} \delta_{r}\left(\eta-\frac{1}{C}\right) C\right] \\
& +\frac{\rho}{2} l^{3} N_{v w N}^{\prime} v w \\
& +\left(x_{G} W-x_{B} B\right) \cos (\theta) \sin (\phi)+\left(y_{G} W-y_{B} B\right) \sin (\theta)
\end{aligned}
$$

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