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ROBUST OPTIMAL SHAPE DESIGN FOR AN ELLIPTIC PDE WITH UNCERTAINTY IN ITS INPUT DATA

JESÚS MARTÍNEZ-FRUTOS¹, MATHIEU KESSLER² AND FRANCISCO PERIAGO²

Abstract. We consider a shape optimization problem for an elliptic partial differential equation with uncertainty in its input data. The design variable enters the lower-order term of the state equation and is modeled through the characteristic function of a measurable subset of the spatial domain. As usual, a measure constraint is imposed on the design variable. In order to compute a robust optimal shape, the objective function involves a weighted sum of both the mean and the variance of the compliance. Since the optimization problem is not convex, a full relaxation of it is first obtained. The relaxed problem is then solved numerically by using a gradient-based optimization algorithm. To this end, the adjoint method is used to compute the continuous gradient of the cost function. Since the variance enters the cost function, the underlying adjoint equation is non-local in the probabilistic space. Both the direct and adjoint equations are solved numerically by using a sparse grid stochastic collocation method. Three numerical experiments in 2D illustrate the theoretical results and show the computational issues which arise when uncertainty is quantified through random fields.

Résumé. Nous considérons un problème d'optimisation de forme pour une équation aux dérivées partielles avec incertitude dans les données d'entrée. La variable de forme apparaît dans le terme de bas degré de l'équation d'état et est modélisée à l'aide d'une fonction indicatrice d'un sous ensemble mesurable du domaine spatial. Comme il est d'usage, une contrainte de mesure est imposée sur la variable de forme. Afin de calculer une forme optimale robuste, la fonction objectif est construite comme une combinaison convexte de la moyenne et la variance de la compliance. Puisque el problème d'optimisation est non convexe, nous en obtenons tout d'abord une relaxation complète. Le problème relaxé est alors résolu numériquement par un algorithme d'optimisation de type gradient. Pour cela, la méthode adjointe est utilisée pour calculer le gradient continu de la fonction coût. Dû à la présence de la variance dans la fonction coût, l'opérateur adjoint associé est non-local dans l'espace de probabilité. Les équations directe et adjointe son résolues numériquement grâce à une méthode de collocation stochastique sur grille sparse. Trois expériences de simulation illustrent certains aspects numériques de la quantification de l'incertidude par champs aléatoires.

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INTRODUCTION

Shape optimization amounts to find the optimal shape of a domain which minimizes or maximizes a given criterion (often called 'objective' or 'cost' function). In view of its many possible engineering applications, it is a widely addressed problem and many aspects of shape optimization are addressed in the literature. We refer for instance to [1, 5, 6, 11, 20, 25] and the references therein.

Probably because of its physical meaning, the compliance is the most studied criterion. Compliance, which is defined as the work done by a load, measures the global rigidity of a physical structure under the action of that load. However, if a shape is optimized for some fixed data (e.g., loads, elastic and/or geometrical material properties, etc..), it is well-known (see for instance [2]) that it could be far from being optimal for small perturbations on that data. Engineering problems are affected by a certain amount of uncertainty in its input data. This uncertainty has several causes (for example, variabilities in the fabrication process of a material) and should be properly quantified. In many applications, Gaussian fields are the model of choice. Uncertainty Quantification (UQ) is at present a very active field of research in Engineering (see [17]), but is not the object of the present paper.

Two standards approaches to include uncertainty in the optimization process are: (1) the worst-case scenario analysis, which leads to a min-max type optimization problem (see for instance [12, 16]), and (2) the so-called average approach, where the optimization problem is described in a probabilistic framework. This paper follows the latter approach.

In the average approach, the statistical moments (mean, variance, etc..) of the objective function are considered in setting up the compliance. For example, Robust Design Optimization (RDO) is concerned with the mean and/or the variance while Reliability-Based Design Optimization (RBDO) deals with higher order statistical properties such as the tails of the probability density function.

It is worth pointing out that most of the engineering-oriented works dealing with shape optimization under uncertainty are based on mathematical programming models. Thus, when the original problem comes from a continuous model (for example, a partial differential equation like the conductivity equation or the elasticity system), a full discretization of all ingredients entering the problem, namely, the underlying state law (typically by using finite elements), the set of admissible shapes and the objective function, is performed at the very beginning. Optimization then applies to the resulting discrete (finite-dimensional) model.

On the contrary, mathematical-oriented papers typically address continuous (infinite-dimensional) optimization problems. A few recent works have for example addressed the issue of shape optimization under uncertainty by using the average approach, see [8,14]. Three common features of these two works are: (1) uncertainty enters in the problem formulation through the loads, (2) only the expected value of the deterministic objective function is considered. Although the expectation gives a first indication of robustness, it is reasonable, in order to achieve a more robust design, to consider as well at least the variance of the objective function. And (3), (approximations of) Gaussian fields as sources of uncertainty in the numerical simulation results presented are not considered. Consequently, the computational complexity involved when Gaussian fields enter into the problem is not shown.

Even if the discrete and continuous methodologies in shape optimization seem at first glance to be equivalent (because even if a continuous model is considered, its numerical resolution necessarily involves a discretization process), they can lead to different results. The subtle difference is that, as proved in some specific optimization problems (see e.g., [10, 18, 29]), the diagrams (a) first discretize and then optimize, and (b) first optimize and then discretize are, in general, not equivalent.

In this paper, we consider a shape optimization problem for an elliptic partial differential equation with uncertainty in both the diffusion coefficient and the load (see Section 1.1). The design variable enters the lower order term of the stochastic equation and is subject to a typical measure constraint which makes the problem non-trivial. The objective function is a weighted sum of the expected value of the compliance and of its variance (Section 1.2). We focus on the numerical resolution of the problem. For that purpose, since the set of admissible shapes is not convex, we follow the relaxation method. Concretely, the relaxed problem involves a simple convexification of the original set of admissible shapes. The existence of a solution for this new problem and the coincidence of the minimum value of the relaxed cost function with the infimum of the original one is established in Theorem 2.1. The numerical resolution of the relaxed problem is then addressed by following a classical (deterministic) gradient-based minimization algorithm. The novelty here is that, due to the presence of the variance in the cost function, the problem becomes non self-adjoint. Consequently, the explicit computation of the continuous gradient (Theorem 3.1) involves the numerical resolution of a Stochastic Elliptic Partial Differential Equation (SEPDE) for the state law and a non-local SEPDE for the adjoint system. We follow a stochastic collocation method, as described in [4], for the numerical resolution of these two problems. To illustrate the theoretical results of this paper, Section 4 describes three numerical experiments where uncertainty appears in the force term (Case 1), in the diffusion coefficient (Case 2), and in the boundary conditions (Case 3).

1. Problem Statement

1.1. The state law

Let $D \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary ∂D , and let (Ω, \mathcal{F}, P) be a complete probability space. If $X \in L^1(\Omega)$ is a real random variable, we denote its expected value by

$$E(X) = \int_{\Omega} X(\omega) \, dP(\omega).$$

Consider the stochastic linear elliptic boundary value problem: find a random function $u: \Omega \times \overline{D} \to \mathbb{R}$ such that *P*-almost surely, the following equation holds

$$\begin{cases} -\nabla \cdot [a(\cdot,\omega)\nabla u(\cdot,\omega)] + 1_{\mathcal{O}}u(\cdot,\omega) = f(\cdot,\omega) & \text{in } D\\ u(\cdot,\omega) = 0 & \text{on } \partial D, \end{cases}$$
(1.1)

where $1_{\mathcal{O}}$ stands for the characteristic function associated to a (Lebesgue) measurable subset $\mathcal{O} \subset D$. Here both the gradient (∇) and the divergence (∇) operators refer to differentiation with respect to the spatial variable $x \in D$.

In order to rewrite (1.1) in a variational form we consider the following Hilbert spaces:

- $\begin{array}{l} \bullet \ L^2_P(D\times\Omega) = \left\{f: D\times\Omega \to \mathbb{R}: \int_D E[f^2] \, dx < \infty\right\}. \\ \bullet \ V_P = L^2_P(\Omega; H^1_0(D)) = \left\{v: D\times\Omega \to \mathbb{R}: \|v\|^2_{V_P} = \int_D E[|\nabla v|^2] \, dx < \infty\right\}. \end{array}$
- $V_{P,a} = \left\{ v \in V_P : \|v\|_{V_{P,a}}^2 = \int_D E[a|\nabla v|^2] \, dx < \infty \right\}.$

Following Babuška et al. [4] we make the following assumptions:

(A1) There exists a random variable $a_{min}(\omega)$ such that

$$a(x,\omega) \ge a_{min}(\omega) > 0$$
 a.s. and a.e. on D . (1.2)

(A2) $f \in L^2_P(D \times \Omega)$.

Notice that condition (1.2) is satisfied with a lognormal diffusion coefficient. For instance, if

$$a(x,\omega) = \exp\left(\sum_{n=1}^{N} b_n(x) Y_n(\omega)\right), \quad Y_n \sim N(0,1) \text{ independent and identically distributed (i.i.d.)},$$

provided that, for all n, b_n is bounded on D.

A second possible specification of the diffusion coefficient, which ensures that condition (1.2) holds is

$$a(x,\omega) - a_0(x) = \exp\left(\sum_{n=1}^N b_n(x)Y_n(\omega)\right), \quad Y_n \sim N(0,1) \text{ independent and identically distributed (i.i.d.)},$$

with $a_0(x) \ge a_0 > 0$ a.e. $x \in D$.

The variational or weak form of problem (1.1) is:

find
$$u \in V_{P,a}$$
 such that $\int_D E[a\nabla u \cdot \nabla v + 1_{\mathcal{O}}uv] dx = \int_D E[fv] dx \quad \forall v \in V_{P,a}.$ (1.3)

As a consequence of assumptions (A1) and (A2) above, the Lax-Milgram lemma ensures the existence and uniqueness of a solution to problem (1.3). Moreover, if $f \in L^2_P(D \times \Omega)$ and $1/a_{min} \in L^2_P(\Omega)$, then $u \in L^2_P(\Omega; H^1_0(D))$ and there exists a constant $C_P > 0$ such that the moment estimates

$$\|u(\cdot,\omega)\|_{H^{1}_{0}(D)}^{2} \leq C_{P} \frac{\|f(\cdot,\omega)\|_{L^{2}(D)}}{a_{min}(\omega)} \quad a.s.$$
(1.4)

and

$$\|u\|_{L_{P}^{2}(\Omega;H_{0}^{1}(D))}^{2} \leq C_{P}\|1/a_{min}\|_{L_{P}^{2}(\Omega)}\|f\|_{L_{P}^{2}(D\times\Omega)}$$
(1.5)

hold (see [4, Lemma 1.2]).

1.2. The optimization problem

The following shape optimization problem with uncertainty is considered:

(OP)
$$\begin{cases} \text{Minimize in } 1_{\mathcal{O}} : \quad J(1_{\mathcal{O}}) = \alpha E\left(\int_{D} f u \, dx\right) + (1 - \alpha) Var\left(\int_{D} f u \, dx\right) \\ \text{subject to} \\ 1_{\mathcal{O}} \in \mathcal{U}_{L} \end{cases}$$

where the set of admissible designs \mathcal{U}_L is given by

$$\mathcal{U}_L = \{ \mathbf{1}_{\mathcal{O}} \in L^{\infty}(D; \{0, 1\}) : |\mathcal{O}| = L|D|, \quad 0 < L < 1 \},$$
(1.6)

 $u = u_{\mathcal{O}}(x, \omega)$ is the associated state, i.e., u solves (1.3),

$$E\left(\int_{D} f u \, dx\right) = \int_{\Omega} \int_{D} f(x,\omega) u(x,\omega) \, dx \, dP(\omega) \tag{1.7}$$

is the expected value of the compliance, and

$$Var\left(\int_{D} fu\,dx\right) = \int_{\Omega} \left(\int_{D} fu\,dx\right)^{2} dP(\omega) - \left(\int_{\Omega} \int_{D} fu\,dx\,dP(\omega)\right)^{2}$$
(1.8)

is its variance. As usual, $|\cdot|$ stands for the Lebesgue measure and $0 \le \alpha \le 1$ is a weighting parameter.

Notice that as a consequence of the estimate (1.5), the cost function in problem (OP) is well-defined.

As indicated in the Introduction, the presence of the variance in the cost function has the effect of increasing the robustness of the optimal design w.r.t. the random perturbations that appear in the state law (1.1).

A physical interpretation of problem (OP) follows: the spatial domain $D \subset \mathbb{R}^d$ could model a string (d = 1) or a membrane (d = 2) which is fixed at the end-points of the string or the boundary of the membrane, and f is a vertical load acting on D. In this case, a = Eh, with E the modulus of elasticity and h the thickness. These input data (also the force f) are taken to be random, which models a certain amount of uncertainty in the material properties (E), in the geometry (h) and in the loads (f). The random output $u(x, \omega)$ then represents the vertical displacement of the point $x \in D$. In problem (OP) we aim at reinforcing a part of the string or the membrane by setting the stiffness of a sub-domain \mathcal{O} of given measure (length in the case of the string and area for the membrane) to 1, in order to maximize the global rigidity of the new structure. In this case, the compliance is the opposite of the rigidity, which implies that, the smaller the compliance, the larger the rigidity. In other words, in the optimization problem (OP) we search for the best (i.e the ones that achieves the minimal compliance) shape and position of the domain \mathcal{O} among all domains which have a fixed measure equal to L|D|.

The deterministic version of problem (OP) was considered for the first time in [19] (see also [20, Chapter 7]). The results in [19] were extended to the case in which the design variable is not a characteristic function but a measure in [7]. More general nonlinear cost functionals were studied in [28].

2. Existence of a solution of the relaxed problem

In order to ensure the existence of a solution for the deterministic version of problem (OP), some sufficient conditions on the source term f and on the domain D are introduced in [19]. However, a general result which states the existence of a solution for this problem is unknown. It is thus quite reasonable to think that the same applies for the probabilistic version considered in this paper. However, even if (OP) admits a solution, it is very convenient to introduce a relaxed version for the numerical resolution of (OP). This is the main goal of this section. For this purpose, we consider the new problem

(ROP)
$$\begin{cases} \text{Minimize in } s: \quad J(s) = \alpha E\left(\int_D f u \, dx\right) + (1 - \alpha) Var\left(\int_D f u \, dx\right) \\ \text{subject to} \\ s \in \overline{\mathcal{U}}_L \end{cases}$$

where now the set of admissible *relaxed* designs $\overline{\mathcal{U}}_L$ is defined as

$$\overline{\mathcal{U}}_L = \left\{ s \in L^{\infty}(D; [0, 1]) : \|s\|_{L^1(D)} = L|D|, \quad 0 < L < 1 \right\}$$
(2.1)

and $u = u_s(x, \omega)$ is the weak solution of problem

find
$$u \in V_{P,a}$$
 such that $\int_D E[a\nabla u \cdot \nabla v + suv] dx = \int_D E[fv] dx \quad \forall v \in V_{P,a}.$ (2.2)

Notice that (ROP) involves a simple convexification of the space of admissible designs for (OP). From now on, we consider the space $L^{\infty}(D; [0, 1])$ endowed with the weak-* topology.

Theorem 2.1. (ROP) is a full relaxation of (OP). Precisely, the functional J(s) is continuous for the weak- \star topology. In particular, there exists $s^{\star} \in \overline{\mathcal{U}}_L$ such that

$$\inf_{1_{\mathcal{O}}\in\mathcal{U}_L}J(1_{\mathcal{O}})=\min_{s\in\overline{\mathcal{U}}_L}J(s)=J(s^*).$$

Moreover, if $\alpha = 1$, then J(s) is convex.

Proof. Let s_n, s be admissible for (ROP) and assume that

$$s_n \rightharpoonup s \quad weak - \star \text{ in } L^{\infty}(D).$$

Denoting by u_n and u the corresponding weak solutions of (2.2), we have that the function $v_n = u_n - u$ solves the problem

$$\begin{cases} -\nabla \cdot [a(\cdot,\omega)\nabla v_n(\cdot,\omega)] + s_n v_n(\cdot,\omega) = (s-s_n)u(\cdot,\omega) & \text{in } D\\ v_n(\cdot,\omega) = 0 & \text{on } \partial D. \end{cases}$$
(2.3)

From (1.4), it is deduced that $v_n(\cdot, \omega)$ is bounded in $H_0^1(D)$ a.s. Thus, by the Rellich theorem, there exists a sub-sequence, still denoted by v_n , and a function $v = v(\cdot, \omega)$ such that

$$v_n \to v$$
 strongly in $L^2(D)$ a.s

As a consequence,

 $uv_n \to uv$ strongly in $L^1(D)$ a.s.

and since $s_n \rightharpoonup s$ weak $- \star$ in $L^{\infty}(D)$, we have that

$$\int_D (s-s_n)uv_n\,dx \to 0 \quad a.s.$$

By dominated convergence,

$$\int_{\Omega} \int_{D} (s - s_n) u v_n \, dx dP(\omega) \to 0.$$
(2.4)

Moreover, from the weak form of (2.3) we have that

$$\int_{\Omega} \int_{D} a |\nabla v_n|^2 \, dx dP(\omega) + \int_{\Omega} \int_{D} s_n v_n^2 \, dx dP(\omega) = \int_{\Omega} \int_{D} (s - s_n) u v_n \, dx dP(\omega).$$

Taking limits in this expression, by (2.4), we conclude that

$$v_n \to 0 \quad \text{in } V_{P,a} \tag{2.5}$$

and also strongly in $H_0^1(D)$ a.s. Since 0 is the single accumulation point of v_n , the whole sequence converges to 0. Taking into account (1.7) and (1.8), the continuity of the term of J(s) corresponding to the expected value of the compliance follows easily. The continuity of the variance term is a direct consequence of the result just proved and of the dominated convergence theorem.

The existence of a solution for problem (ROP) is a consequence of the continuity of J(s) and of the compactness of $\overline{\mathcal{U}}_L$ w.r.t. the weak-* topology. By density, the infimum of $J(1_{\mathcal{O}})$ in \mathcal{U}_L equals the minimum of J(s) in $\overline{\mathcal{U}}_L$.

Let us now take $\alpha = 1$, i.e., the cost function J(s) only involves the expectation of the compliance. Since u_s achieves the minimum of the functional

$$\frac{1}{2}\int_{\Omega}\int_{D}\left[a|\nabla v|^{2}+sv^{2}\right]\,dxdP(\omega)-\int_{\Omega}\int_{D}fv\,dxdP(\omega),$$

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by (2.2),

$$\begin{array}{ll} -\frac{1}{2}J(s) &= -\frac{1}{2}\int_{\Omega}\int_{D}\left[a|\nabla u|^{2} + su^{2}\right]\,dxdP(\omega) \\ &= \min_{v\in V_{P,a}}\left\{\frac{1}{2}\int_{\Omega}\int_{D}\left[a|\nabla v|^{2} + sv^{2}\right]\,dxdP(\omega) - \int_{\Omega}\int_{D}fv\,dxdP(\omega)\right\}, \end{array}$$

which is concave as a minimum of affine functions. This proves that J(s), with $\alpha = 1$, is convex.

3. Numerical resolution of the relaxed problem

For the numerical resolution of problem (ROP) we use a gradient-based optimization algorithm. Precisely, we shall use the well-known (in structural optimization) Method of Moving Asymptotes (MMA) as described in [26, 27]. We first focus on the numerical computation of the continuous gradient of the cost function in problem (ROP).

3.1. Explicit computation of the continuous gradient

Theorem 3.1. The functional J(s) is Gâteaux differentiable at each admissible $s \in \overline{\mathcal{U}}_L$. Moreover, its derivative at s is given by

$$J'(s)(x) = -\alpha \int_{\Omega} u_s^2(x,\omega) dP(\omega) + (1-\alpha) \int_{\Omega} u_s(x,\omega) p_s(x,\omega) dP(\omega) + 2(1-\alpha) \left(\int_{\Omega} \int_D f u_s dx dP(\omega) \right) \int_{\Omega} u_s^2(x,\omega) dP(\omega)$$
(3.1)

where u_s solves the direct problem (2.2) and the adjoint state p_s is the solution of the following non-local stochastic elliptic problem: find $p_s \in V_{P,a}$ such that

$$\int_{\Omega} \int_{D} \left[a \nabla p_s \cdot \nabla v + s p_s v \right] \, dx dP(\omega) = -2 \int_{\Omega} \left[\int_{D} f u_s \, dx \int_{D} f v \, dx \right] \, dP(\omega) \quad \forall v \in V_{P,a}. \tag{3.2}$$

Proof. Let $\hat{s} \in L^{\infty}(D; [0, 1])$ be an admissible direction, i.e., for $\varepsilon > 0$ small enough, $s + \varepsilon \hat{s} \in \overline{\mathcal{U}}_L$. We first consider the term (1.7) in J(s) corresponding to the expected value of the compliance. To begin with, we notice that the solution $u_{s+\varepsilon \hat{s}}$ associated to the perturbation $s + \varepsilon \hat{s}$ can be decomposed as $u_{s+\varepsilon \hat{s}} = u_s + \hat{u}_{\varepsilon}$, where \hat{u}_{ε} solves the boundary value problem

$$\begin{cases} -\nabla \cdot (a\nabla \hat{u}_{\varepsilon}) + (s + \varepsilon \hat{s})\hat{u}_{\varepsilon} = -\varepsilon \hat{s}u_s & \text{in } D \times \Omega \\ \hat{u}_{\varepsilon} = 0 & \text{on } \partial D \times \Omega. \end{cases}$$
(3.3)

Moreover, thanks to the moment estimate (1.5),

$$\|\hat{u}_{\varepsilon}\|^{2}_{L^{2}_{P}(\Omega;H^{1}_{0}(D))} \leq \varepsilon C, \qquad (3.4)$$

where C > 0 depends on the L_P^2 -norm of $1/a_{min}$ and f. Subtracting the equations for $s + \varepsilon \hat{s}$ and for s, multiplying the result by $(u_{s+\varepsilon \hat{s}} + u_s)$, and finally integrating over Ω and D we get

$$\begin{split} &\int_{\Omega} \int_{D} \left[a |\nabla u_{s+\varepsilon \hat{s}}|^{2} + (s+\varepsilon \hat{s}) u_{s+\varepsilon \hat{s}}^{2} \right] \, dx dP(\omega) - \int_{\Omega} \int_{D} \left[a |\nabla u_{s}|^{2} + s u_{s}^{2} \right] \, dx dP(\omega) \\ &= -\varepsilon \int_{\Omega} \int_{D} \hat{s} u_{s}^{2} \, dx dP(\omega) - \varepsilon \int_{\Omega} \int_{D} \hat{s} \hat{u}_{\varepsilon} u_{s} \, dx dP(\omega). \end{split}$$

Therefore,

$$\frac{E\left(\int_{D} f u_{s+\varepsilon \hat{s}} \, dx\right) - E\left(\int_{D} f u_{s}\right)}{\varepsilon} = -\int_{\Omega} \int_{D} \hat{s} u_{s}^{2} \, dx dP(\omega) - \int_{\Omega} \int_{D} \hat{s} \hat{u}_{\varepsilon} u_{s} \, dx dP(\omega)$$

 \Box

Applying (1.5) to u_s and considering (3.4), a direct application of Holder's inequality shows that the second term in the right-hand side of this expression vanishes as $\varepsilon \to 0$. The first term in the right-hand side of (3.1) is then derived.

The third term in (3.1) comes from the second term in (1.8) just as a straightforward application of the chain rule and the result just proved.

Finally, let us compute the second term in the right-hand side of (3.1). Using again the decomposition $u_{s+\varepsilon\hat{s}} = u_s + \hat{u}_{\varepsilon}$, we have

$$\int_{\Omega} \left(\int_{D} f u_{s+\varepsilon \hat{s}} \, dx \right)^{2} dP(\omega) - \int_{\Omega} \left(\int_{D} f u_{s} \, dx \right)^{2} dP(\omega) = \int_{\Omega} \left(\int_{D} f \hat{u}_{\varepsilon} \, dx \right)^{2} dP(\omega) + 2 \int_{\Omega} \left(\int_{D} f u_{s} \, dx \int_{D} f \hat{u}_{\varepsilon} \, dx \right) dP(\omega).$$

By (1.4), the first term in the right-hand side of this expression behaves as ε^2 and hence

$$\lim_{\varepsilon \to 0} \frac{\int_{\Omega} \left(\int_{D} f u_{s+\varepsilon \hat{s}} \, dx \right)^2 dP(\omega) - \int_{\Omega} \left(\int_{D} f u_s \, dx \right)^2 dP(\omega)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{2 \int_{\Omega} \left(\int_{D} f u_s \, dx \int_{D} f \hat{u}_{\varepsilon} \, dx \right) dP(\omega)}{\varepsilon}.$$
 (3.5)

On the other hand, putting $v = \hat{u}_{\varepsilon}$ in (3.2) yields

$$2\int_{\Omega} \left(\int_{D} f u_{s} \, dx \int_{D} f \hat{u}_{\varepsilon} \, dx \right) dP(\omega) = -\int_{\Omega} \int_{D} \left[a \nabla p_{s} \cdot \nabla \hat{u}_{\varepsilon} + sp_{s} \hat{u}_{\varepsilon} \right] \, dx dP(\omega) = \int_{\Omega} \int_{D} \varepsilon \hat{s} u_{s} p_{s} \, dx dP(\omega) + \int_{\Omega} \int_{D} \varepsilon \hat{s} \hat{u}_{\varepsilon} p_{s} \, dx dP(\omega),$$

the second equality being a consequence of the variational formulation of (3.3). We now are able to compute the limit in (3.5) and, thanks to the estimate (3.4), obtain the desired result.

From (3.1) it is deduced that the gradient of the cost function depends on the solution of the direct problem (2.2), on the solution of its associated adjoint equation (3.2), and on the corresponding integrals over the probability space. We now describe how to solve numerically these three problems.

3.2. Numerical resolution of the state and adjoint state equations, and numerical approximation of the related statistical quantities.

Throughout this section we follow [4] and refer to it for more details.

Following [4] we make the following assumption, which is not too restrictive since in many problems the source of randomness can be approximated by a finite number of random variables (see, for instance, the case of a truncated Karhunen-Loève expansion [21] [22]).

ASSUMPTION 3.1 (Finite dimensional noise). The random input data take the form:

$$a(x,\omega) = a(x, Y_1(\omega), \cdots, Y_N(\omega))$$
 and $f(x,\omega) = f(x, Y_1(\omega), \cdots, Y_N(\omega))$

where $\{Y_n\}_{n=1}^N$ are real-valued random variables with mean value zero and unit variance.

Denote by $\Gamma_n \equiv Y_n(\Omega)$ the image of Y_n , $\Gamma = \prod_{n=1}^N \Gamma_n$ and assume that $[Y_1, \dots, Y_N]$ have a joint probability density function $\rho : \Gamma \to \mathbb{R}_+$, with $\rho \in L^{\infty}(\Gamma)$.

3.2.1. The direct problem

For all $x, \omega \mapsto u(x, \omega)$ is measurable with respect to \mathcal{F} . As a consequence of Assumption 3.1, the Doob-Dynkin's lemma therefore ensures that the solution $u(x, \omega)$ of (2.2) can be represented as

$$u(x,\omega) = u(x, Y_1(\omega), \cdots, Y_N(\omega)).$$

The stochastic boundary value problem (2.2) thus becomes the following deterministic Dirichlet boundary value problem for an elliptic PDE with an N-dimensional parameter:

find
$$u \in V_{\rho,a}$$
 such that $\int_{\Gamma} \rho[(a\nabla u, \nabla v)_{L^2(D)} + s(u, v)_{L^2(D)}] dy = \int_{\Gamma} \rho(f, v)_{L^2(D)} dy, \quad \forall v \in V_{\rho,a}.$ (3.6)

Here $(\cdot, \cdot)_{L^2(D)}$ stands for the inner product in $L^2(D)$ and $V_{\rho,a}$ is the analogue of $V_{P,a}$ with (Ω, \mathcal{F}, P) replaced by $(\Gamma, \mathcal{B}^N, \rho dy)$, with \mathcal{B}^N the σ -algebra of the Borel sets in \mathbb{R}^N . This problem is equivalent to

$$\int_{D} [a(y)\nabla u \cdot \nabla \phi + su\phi] \, dx = \int_{D} f(y)\phi \, dx, \quad \forall \phi \in H^{1}_{0}(D), \quad \rho - a.e. \text{ in } \Gamma.$$
(3.7)

Assuming that both a and f admit a smooth extension on the ρdy -zero sets, (3.7) can be extended a.e. in Γ w.r.t. the Lebesgue measure.

To approximate numerically the solution of (3.6), the spaces are considered $V_{p,h} = \mathcal{P}_p(\Gamma) \bigotimes H_h(D)$, where:

- $H_h(D) \subset H_0^1(D)$ is a standard finite element approximation of $H_0^1(D)$, and
- $\mathcal{P}_p(\Gamma) \subset L^2_\rho(\Gamma)$ is the span of tensor product of suitable orthogonal polynomials with degree at most $p = (p_1, \cdots, p_N)$, i.e., $\mathcal{P}_p(\Gamma) = \bigotimes \mathcal{P}_{p_n}(\Gamma_n)$, with $\mathcal{P}_{p_n}(\Gamma_n) = \operatorname{span}(y_n^m, m = 0, \cdots, p_n)$, $n = 1, \cdots, N$.

The numerical algorithm to solve (3.6) consists of three main steps: Step 1. Project equation (3.6) onto $H_h(D)$:

$$\int_{D} [a(y)\nabla u_h \cdot \nabla \phi_h + su_h \phi_h] \, dx = \int_{D} f(y)\phi_h \, dx, \forall \phi_h \in H_h(D), a.e. \ y \in \Gamma.$$

Step 2. Collocate the above equation at the zeros y_k of the corresponding orthogonal polynomials.

Step 3. Build the discrete solution $u_{h,p} \in \mathcal{P}_p(\Gamma) \bigotimes H_h(D)$ by interpolating in y the collocated solutions.

3.2.2. The adjoint problem

To solve the adjoint problem (3.2) we follow the same lines as for the direct problem just described. In this case, the main difference w.r.t. the direct problem refers to the non-local term that appears in the right-hand side of (3.2). We next explain how to overcome this difficulty.

First, the continuous equation (3.2) is projected onto the discrete space $H_h(D)$. Let us denote by $p_h(y)$ the solution of this equation. Second, we collocate the resulting equation at the zeros y_k of the orthogonal polynomials of $\mathcal{P}_p(\Gamma)$. Then, at each node y_k of the probability space, we get

$$\int_D [a(y_k)\nabla p_h \cdot \nabla \phi_h + sp_h\phi_h] \, dx = \left(-2\int_D f(y_k)u_h(y_k) \, dx\right) \int_D f(y_k)\phi_h \, dx.$$

Hence,

$$p_h(y_k) = c_k u_h(y_k), \text{ with } c_k = -2 \int_D f(y_k) u_h(y_k) \, dx.$$

3.2.3. Numerical approximation of integrals over the probability space

As indicated above, to compute the gradient of the cost function and also to compute the cost function of problem (ROP) itself, it is required to approximate numerically integrals over the space Γ . More precisely, given a continuous function $g: \Gamma \to \mathbb{R}$ we use a Gauss quadrature formula approximating $\int_{\Gamma} g(y)\hat{\rho}(y) dy$. Here $\hat{\rho}$ is an auxiliary probability density function that satisfies

$$\hat{\rho}(y_1, \cdots, y_n) = \prod_{n=1}^N \hat{\rho}_n(y_n) \text{ and } \|\frac{\rho}{\hat{\rho}}\|_{L^{\infty}(\Gamma)} < \infty.$$

We refer to [4] for more details. Denoting by $y_k = (y_{1,k_1}, y_{2,k_2}, \cdots, y_{N,k_N})$ a point in Γ , we introduce, for $n = 1, \cdots, N$, the Lagrange basis $\{l_{n,j}\}_{j=1}^{p_n}$ of the space \mathcal{P}_{p_n} , i.e.,

$$l_{n,j} \in \mathcal{P}_{p_n}(\Gamma_n), \quad l_{n,j}(y_{n,k}) = \delta_{jk}, \quad j,k = 1, \cdots, p_n + 1$$

where δ_{jk} is the Kronecker symbol. Finally, we set $l_k(y) = \prod_{n=1}^N l_{n,k_n}(y_n)$. With this notation, the Gauss quadrature formula $E^p_{\hat{\rho}}[g]$ approximating $\int_{\Gamma} g(y)\hat{\rho}(y) \, dy$ is given by

$$E_{\hat{\rho}}^{p}[g] = \sum_{k=1}^{N} \omega_{k} g(y_{k}), \quad \omega_{k} = \prod_{n=1}^{N} \omega_{k_{n}}, \quad \omega_{k_{n}} \int_{\Gamma_{n}} l_{k_{n}}^{2}(y) \hat{\rho}(y) \, dy.$$
(3.8)

4. Numerical results

Next, we consider three numerical examples for problem (ROP). These examples have been studied previously from a deterministic point of view in [19]. For comparison reasons, we slightly adapt those examples in order to deal with uncertainties in the force term (Case 1), in the diffusion coefficient (Case 2), and in the boundary conditions (Case 3).

The numerical experiments treated in this section are characterized by the following common features:

- Finite element approximation in the physical domain. A stochastic collocation method is used to numerically solve both the direct and adjoint PDE, as well as to perform the integration required to consider the statistical moments of the compliance. The stochastic collocation method requires the resolution of a set of uncoupled deterministic sub-problems in several nodes in the stochastic domain. Each sub-problem is discretized in the physical domain through Finite Elements, using P1 elements on a triangular mesh. The maximum allowed element edge length is fixed to a maximum of h = 0.01 in all the numerical experiments.
- Uncertainty modeling. Two random fields are used to model uncertainty in the first two experiments. In order to satisfy the ellipticity condition (1.2), a lognormal random field is selected for the diffusion coefficient. Consider

$$F(x) = \exp\left(\eta(x) + \xi(x) U(x)\right), \tag{4.1}$$

where $\eta(x)$ and $\xi(x)$ are the location and scale parameters of the lognormal distribution, that may depend on $x \in D$. The Gaussian random field U is assumed to have a zero mean, a unit variance and the following isotropic squared exponential correlation function:

$$C(x, x') = exp\left[-\sum_{i=1}^{2} \frac{(x_i - x'_i)^2}{l^2}\right],$$
(4.2)

where $x = (x_1, x_2), x' = (x'_1, x'_2) \in D$, and l > 0 is the correlation length. The Gaussian random field U is discretized through the truncated Karhunen-Loève expansion [21] [22]:

$$U(x) = \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega), \qquad (4.3)$$

where λ_n and b_n are, respectively, the eigenvalues and eigenvectors of the compact and self-adjoint operator

$$\psi \mapsto \int_D C(x, x')\psi(x') \, dx', \quad \psi \in L^2(D),$$

and $Y_n(\omega)$ is a vector of independent normal random variables with zero mean and unit variance.

In this work, the LU triangular decomposition of the covariance matrix proposed by [15] is used in order to generate the random fields simulations. The decay rate of the eigenvalues is analyzed and the size N of the expansion is fixed in order to capture a 95% of the energy field. More precisely, we set N to be the smallest positive integer such that

$$\frac{\sum_{n=1}^{N} \lambda_n^h}{\operatorname{trace}(C_h)} \ge 0.95,\tag{4.4}$$

where λ_n^h are the numerical approximation of the eigenvalues considered in (4.3), ordered in nonincreasing order, and trace(C_h) is the trace of the discrete (over the spatial discretization of D) correlation function given in (4.2). The parameter h stands for the size of the mesh in the spatial domain. Uncertainty in the third numerical example is fully characterized by one single normal random variable.

• Collocation approximation in stochastic domain. Since the random variables Y_n come from a Karhunen-Loève expansion, they are independent standard Gaussian. Hence, the corresponding collocation nodes y_k are the Cartesian products determined by the roots of Hermite polynomials. In order to reduce the computational cost related to the full tensor product rules, an isotropic sparse grid, as proposed in [24], has been applied. Given a 1D quadrature rule Q^k of level k, the M-dimensional Smolyak isotropic product rule is defined as follows:

$$\mathcal{A}(k,M) = \sum_{k-M+1 \le |i| \le k} (-1)^{k+M-|i|} \binom{k+M}{k+M-|i|} \left(\mathcal{Q}^{i_1} \otimes \cdots \otimes \mathcal{Q}^{i_M} \right).$$
(4.5)

The dimension of the product rule is equal to the size N of the Karhunen-Loève expansion. The nonintrusive characteristic of the stochastic collocation method allows to carry out the parallel computation of the deterministic solver at all of the collocation nodes.

• Optimization algorithm. Our robust optimal shape design problem (OP) aims to find a trade-off between optimality and robustness. Actually, the problem should be formulated as a multi-objective optimization problem encompassing two objective functions: the mean and the variance of the compliance. This optimization problem yields a set of optimal solutions in the Pareto sense [23]. In the present work, the set of Pareto optimal solutions is obtained by a previous scalarization of the multi-objective problem into a single-objective one using a weighted-sum approach. To ensure the same order of magnitude [13], the mean and the variance of the compliance are scaled using the following strategy:

$$J(s) = \alpha \frac{E\left(\int_D f u \, dx\right)}{E^*} + (1-\alpha) \frac{Var\left(\int_D f u \, dx\right)}{V^*}$$

$$\tag{4.6}$$

where $\alpha \in [0, 1]$ is the weight balancing the influence of each objective functional. E^* and V^* are the mean and the variance of the compliance obtained for the initial design which are used as scale factors. The Method of Moving Asymptotes (MMA), developed by Svanberg in [26] [27] is used to solve the optimization problem. The MMA solves a sequence of sub-problems generated from local convex approximations of the objective function and its constraints, based on their values and sensitivities. Using the MMA our problem can be reformulated as follows:

$$\begin{cases}
\text{Minimize in } s = (s_1, \cdots, s_n): \quad J(s) + a_0 z + cy + \frac{1}{2} dy^2 \\
\text{subject to} & \text{PDE constraint,} \\ & \frac{\|s\|_{L^1(D)}}{L|D|} - 1 - az - y \le 0 \\
& 0 \le s_j \le 1 \quad j = 1, \dots, n \\
& z \ge 0, \quad y \ge 0,
\end{cases}$$
(4.7)

where n is the number of nodes used to discretize the design variable $s, y \in \mathbb{R}$ and $z \in \mathbb{R}$ are "artificial" variables [27], while a_0, a, c , and d are given real numbers such that $a_0 > 0$, a > 0, c > 0, $d \ge 0$, and c + d > 0. As indicated above, the PDE constraint in problem (ROP) has been discretized and solved by using finite elements in the physical domain and a stochastic sparse grid in the random domain. Following Svanberg's suggestion [27], the values $a_0 = 1$, a = 0, c = 1000 and d = 0 are selected for all the numerical experiments. The maximum number of iterations is set to 300, and the tolerance of the objective function, constraint, and variables is set to 1e-6. We refer the reader to [26] [27] for a more detailed description of the algorithm.

4.1. Case 1: Uncertainty in the force term

The computational domain is a two dimensional unit square $D =]0, 1[^2$. The diffusion coefficient a = 1 and the force term $f(x, y, \omega)$ is the piece-wise function:

$$f(x, y, \omega) = \begin{cases} 20 & 0 \le x \le 0.4, 0 \le y \le 1, \\ 0.001 & 0.4 < x \le 0.6, 0 \le y \le 1, \\ f_2(x, y, \omega) & 0.6 < x \le 1, 0 \le y \le 1, \end{cases}$$
(4.8)

where $f_2(x, y, \omega)$ is a lognormal random field obtained through the transformations (4.1)-(4.3) with $\eta = 2.9761$ and $\xi = 0.1980$. An isotropic correlation length l = 0.3 is considered. The size of the Karhunen-Loève expansion is N = 10 and captures the 95% of the energy field. The level of the 1D quadrature rules is k = 3. Using P1 finite elements in space and an isotropic sparse grid collocation approximation with Gauss-Hermite collocation nodes, the solution $u(x, \omega)$ is computed on each finite element node of 11662 nodes in D and each collocation node of 221 in Ω . The expectation and standard deviation of the uncontrolled (i.e., s = 0) solution $u(x, \omega)$ are represented in Figure 1. Due to the uncertainty introduced in the force term, the region $\{0.6 \le x \le 1\}$ presents the largest standard deviation.

The optimization problem (ROP) has been solved for values of the weight factor $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 0.85, 0.9, 0.95$ and 1. The measure constraint is $|\mathcal{O}| = 0.1|D|$. In order to analyze the effectiveness of the analytical expression for the sensitivities of the statistical moments, the convergence history of the optimization algorithm, for $\alpha = 0$, is plotted in Figure 2. Notice that the objective function presents a peak after the first iteration due to the in-feasibility of the chosen initial design. Once the algorithm has achieved a feasible design, the objective function exponentially decreases until convergence (iteration 53). Figures 3 and 4 show the Pareto front of optimal solutions and the optimal designs s(x, y), respectively. The mean value and variance of the solution $u(x, y, \omega)$ associated with the deterministic optimal solution (which is plotted in Figure 4 (f)) have been computed and plotted along the Pareto front of optimal solutions (see star in Figure 3).

The optimal solutions for the robust shape optimization problem can be compared to the deterministic one. In this example, the optimal solution for the case of minimal expectation is very close to the deterministic solution (Figure 4 (e)-(f)). As expected, for the most robust design (minimal variance) all the reinforcement

is devoted to strengthen the region of maximum variability of the solution (see Figure 1 (b) and Figure 4 (a)). When only the expectation of the compliance is considered, the optimal design (Figure 4 (e)) adopts the region where the displacement is the largest (Figure 1 (a)). The solutions of the optimization problems for intermediate values of the weight parameter $\alpha = 0.2, 0.4, 0.6, 0.8, 0.85, 0.9$ and 0.95 yield a trade-off between robustness and performance (Figure 4), providing a designer with a set of compromise solutions.

4.2. Case 2: Uncertainty in the diffusion coefficient

As in Case 1, the physical domain is the unit square $D =]0, 1[^2$. In order to produce more realistic results, the lower-order term $s(x, y)u(x, y, \omega)$ in (2.2) has been replaced by $bs(x, y)u(x, y, \omega)$, with b = 100. The force term is $f(x, y) = 100\sin(3\pi x)\sin(5\pi y)$, and the diffusion coefficient $a(x, y, \omega)$ is a lognormal random field obtained through the transformations (4.1)-(4.3) with

$$\eta(x,y) = \log\left(\frac{1}{(1+\Psi(x,y))^{1/2}}\right),$$

$$\xi(x,y) = (\log(1+\Psi(x,y)))^{1/2},$$

$$\Psi(x,y) = \frac{256}{25}(-x^2+x)^2(-y^2+y)^2.$$
(4.9)

An isotropic correlation length l = 0.3 is considered. The size of the Karhunen-Loève expansion is N = 14 capturing the 95% of the energy field. The level of the 1D quadrature rules is k = 4.

Figure 5 plots the expectation and standard deviation of the uncontrolled (s = 0) state variable $u(x, \omega)$. The parameter for the measure constraint L = 0.1. Seven single-objective optimization problems are derived from the multi-objective scalarization by modifying the weight factor $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 0.9, 1$. The convergence of the objective function and the evolution of the constraint fulfillment is similar to Case 1. The solution of the single-objective problems yield different optimal relaxed shapes, which are shown in Figure 7 over the physical domain and in Figure 6 over the objective function domain. Similarly to Case 1, as shown in Figure 7, the minimal expectation design is very close to the deterministic one. However, the design that provides more robustness (minimal variance) consists of a reinforcement in the region of the domain with the largest variability in the diffusion coefficient.

4.3. Case 3: Uncertainty in the boundary conditions

The goal of this numerical experiment is to obtain a reinforcement configuration of an elastic membrane less sensitive to modifications in the boundary conditions and with a better performance. Concretely, we consider the following boundary value problem with uncertainty in the boundary conditions:

$$\begin{cases}
-\nabla \cdot \nabla u(\cdot,\omega) + s(\cdot)u(\cdot,\omega) = 1 & \text{in } D \\
u(\cdot,\omega) = 0 & \text{on } \Gamma_D(\omega), \\
\frac{\partial u}{\partial \vec{n}}(\cdot,\omega) = 0 & \text{on } \Gamma_N(\omega),
\end{cases}$$
(4.10)

where the physical domain is the unit circle $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and \vec{n} denotes the unit outward normal vector. The boundary of D is decomposed into two disjoint random parts $\Gamma = \Gamma_D(\omega) \cup \Gamma_N(\omega)$, with $\Gamma_D(\omega) \cap \Gamma_N(\omega) = \emptyset$. $\Gamma_D(\omega)$ is parametrized as

$$\gamma_D(\omega) : \left[\frac{\omega}{2}, \frac{\pi}{2} - \frac{\omega}{2}\right] \bigcup \left[\frac{\pi}{2} + \frac{\omega}{2}, \pi - \frac{\omega}{2}\right] \bigcup \left[\pi + \frac{\omega}{2}, \frac{3\pi}{2} - \frac{\omega}{2}\right] \bigcup \left[\frac{3\pi}{2} + \frac{\omega}{2}, 2\pi - \frac{\omega}{2}\right] \longrightarrow \mathbb{R}^2$$

$$\theta \longmapsto \gamma_D(\theta) = (\cos\theta, \sin\theta)$$

$$(4.11)$$

where ω is a random variable that follows a truncated normal distribution with zero mean and $\sigma = \pi/12 \ rad$ standard deviation. The lower/upper truncation point of the distribution is $-2\sigma/+2\sigma$. The variational form of problem (4.10) is: find $u \in L^2_P(\Omega; H^1(D))$ such that

$$\int_D E[\nabla u \cdot \nabla v + suv] \, dx = \int_D E[fv] \, dx \quad \forall v \in L^2_P(\Omega; H^1(D)).$$

and the boundary conditions are satisfied a.s. $\omega \in \Omega$. Precisely,

 $u(\cdot, \omega) = 0$ on $\Gamma_D(\omega)$ in the sense of traces and a.s. $\omega \in \Omega$,

and $\frac{\partial u}{\partial \vec{n}}(\cdot,\omega) = 0$, a.s. $\omega \in \Omega$, as an element of $H^{-1/2}(\Gamma_N(\omega))$, i.e.

$$< \frac{\partial u}{\partial \vec{n}}, \phi >= 0 \quad \forall \phi \in H^{1/2}(\Gamma_N(\omega)).$$

We notice that, with almost no changes, the theoretical results in sections 2 and 3 easily extend to the case of problem (4.10).

In the numerical implementation of this experiment the level of the 1D quadrature rule is k = 4.

Contrary to the previous cases, in this experiment the minimal expectation design differs slightly from the deterministic one. This difference can be explained by the qualitative change in the contours of the solution of problem (4.10) (Figure 9(a)) in comparison to the deterministic solution. This change is related to the type of uncertainty introduced into the problem. Figure 9(b) presents the standard deviation of the uncontrolled (s = 0) state variable $u(x, \omega)$, showing the maximum deviation at $\theta = 0, \pi/2, \pi, 3\pi/2$ rad.

The parameter for the measure constraint is L = 0.25. Five single-objective optimization problems are derived from the multi-objective scalarization by modifying the weight factor $\alpha = 0, 0.25, 0.5, 0.75, 1$. The convergence history of the algorithm is similar to the two previous examples. The Pareto front is depicted in Figure 10. As in the previous cases, the weighted-sum approach allows us effectively approximate the convex Pareto front of optimal solutions. Each solution represents a different configuration of the membrane reinforcement as shown in Figure 8. The most robust design (Figure 8(a)) balances the reinforcement between the central region of the domain (Figure 9 (a)) and the region with a largest deviation of the compliance (Figure 9 (b)). However, the design with minimal expectation of the compliance concentrates all the allowed reinforcement over a region close to a circle centered at the origin and with radius 0.5 (solution of the deterministic problem).

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FIGURE 1. Case 1. Expectation (a) and standard deviation (b) of $u(x, y, \omega)$ without optimization, i.e. s = 0.



FIGURE 2. Case 1. Convergence history of the optimization algorithm for $\alpha = 0$. Objective function (a) and measure constraint (b).



FIGURE 3. Case 1. Pareto front of optimal solutions (circles) and deterministic optimal solution (star). From left to right, and top to bottom, the circles correspond to the values $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 0.85, 0.9, 0.95$ and 1.



FIGURE 4. Case 1. Optimal design s(x, y) for different values of α . Case (a) corresponds to minimal variance, case (e) minimal expectation, and case (f) optimal deterministic design.



FIGURE 5. Case 2. Expectation (a) and standard deviation (b) of $u(x, y, \omega)$ without optimization (s=0).



FIGURE 6. Case 2: Pareto front of optimal solutions (circles) and deterministic optimal solution (star). From left to right, and top to bottom, the circles correspond to the values $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 0.9$ and 1.



FIGURE 7. Case 2. Optimal design s(x, y) for different values of α . Case (a) corresponds to minimal variance, case (e) to minimal expectation, and case (f) is the optimal deterministic design.



FIGURE 8. Case 3. Optimal design s(x, y) for different values of α . Case (a) corresponds to minimal variance, case (e) to minimal expectation, and case (f) is the optimal deterministic design.



FIGURE 9. Case 3: Expectation (a) and standard deviation (b) of the uncontrolled (s = 0) solution.



FIGURE 10. Case 3. Pareto front of optimal solutions (circles) and deterministic solution (star). From left to right, and top to bottom, the circles correspond to the values $\alpha = 0, 0.25, 0.5, 0.75$ and 1.