# Optimal design of the time-dependent support of bang-bang type controls for the approximate controllability of the heat equation 

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#### Abstract

We consider the nonlinear optimal shape design problem which consists in minimizing the amplitude of bang-bang type controls for the approximate controllability of a linear heat equation with a bounded potential. The design variable is the time-dependent support of the control. As usual, a volume constraint is imposed on the design variable. Thus, we look for the best space-time shape and location of the support of the control among those which have the same Lebesgue measure. Since the admissibility set for the problem is not convex, we first obtain a well-posed relaxation of the original problem and then use it to derive a descent method for the numerical resolution of the problem. Numerical experiments in 2D seem to indicate that, even for a regular initial datum, the original problem does not have a solution and therefore a true relaxation phenomenon occurs in this context. Also, we implement a simple algorithm for computing a quasi-optimal domain for the original problem from the optimal solution of its associated relaxed one.


Keywords Linear heat equation with potential • approximate controllability • bang-bang control • optimal design of support • relaxation • numerical approximation.

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## 1 Introduction

### 1.1 Motivation

The problem of optimizing some parameters such as size, position or number of sensors and/or actuators in control systems modelled by partial differential equations has attracted the interest of the engineering community during the last decades (see for instance [8, 13 and the references therein).

On the other hand, the mathematical community has addressed mainly the problem of determining the optimal shape and position of the region where controls act in a distributed parameter system. One of the main differences with respect to engineering-oriented papers is that no restrictions on the shape of the admissible regions are imposed, that is, an admissible domain is just a measurable set which satisfies a certain volume constraint. This point of view implies that, very often, the optimal design problem does not have a solution. Thus, the relaxation method has became one of the most popular techniques to deal with this type of optimization problems. Indeed, in the context of elliptic PDEs, in 11 the problem was studied of optimal reinforcing a part of a membrane in order to minimize the work made by an a priori fixed load. Nonexistence of a solution was proved for generic loads. For the case of hyperbolic equations, in [9, 10 the authors consider the problem of optimal shape and position of the actuators for the stabilization of the wave equation. An interesting point of these two works is that the optimality criterium does not depend on the initial conditions of the underlying PDE. The case of optimal design of the support for the exact controllability problem of the one-dimensional wave equation has been recently addressed for the case of fixed initial data 15,19 , 20 and also uniformly with respect to the initial conditions [20]. A numerical study of the same problem in 2D may be found in [14]. Even for the simplest case in which the initial data are fixed, the optimal domain, if it exist, may have a very complicated structure. Precisely, it is proved in [20] that for the case of exact controls with minimal $L^{2}$-norm there are initial data of class $C^{\infty}$ for which the corresponding optimal domain is of fractal type. Consequently, if the optimal support of the control may be so complicated, then it is natural to simplify the way in which the control acts on the system. Typically, piecewise constant (in particular, bang-bang) controls are very suitable to do this task. This is a first motivation for this work. Second, we would like to consider the case in which the support of the control is not fixed, i.e., it changes with time. This situation was suggested in 18 but, up to best knowledge of the author, it has not been studied so far. Finally, this paper aims at proposing a simple way to extend to higher dimensions and to the case of optimal controls in the $L^{\infty}$-norm (at least at the level of numerical simulation) some of the
above-mentioned works, in particular [16, which were limited to one-space dimension and to the $L^{2}$-case.

### 1.2 Problem formulation

The present paper is concerned with the optimal design of the time-dependent shape and position of support of bang-bang type controls for the approximate null controllability of the heat equation. Let us now state the problem.

First, consider the problem of minimizing the amplitude of the bang-bang control for the approximate controllability of the heat equation. By using a penalty approach for the approximate null controllability condition, the problem takes the form

$$
\left\{\begin{array}{lll}
\text { Minimize in }\left(\lambda, 1_{\mathcal{O}}\right): & \frac{1}{2}\left(\lambda^{2}+\frac{1}{\alpha}\|y(T)\|_{L^{2}(\Omega)}^{2}\right) & \\
\text { subject to } & & \\
& y_{t}-\Delta y+a y=\left[\lambda\left(21_{\mathcal{O}}-1\right)\right] 1_{q}, & (x, t) \in Q \\
y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\
& y(x, 0)=y_{0}(x), & x \in \Omega \\
& \left(\lambda, 1_{\mathcal{O}}\right) \in \mathbb{R}^{+} \times L^{\infty}(q ;\{0,1\}) &
\end{array}\right.
$$

where $\Omega$ is an open and bounded set of $\mathbb{R}^{N}, N \geq 1$, with $C^{2}$ boundary $\Gamma$, $Q=\Omega \times(0, T)$, with $T>0, \Sigma=\Gamma \times(0, T), q \subset \subset Q$ and $\mathcal{O} \subset \subset Q$ are (small) non-empty measurable sets of $Q, 1_{q}=1_{q}(x, t)$ and $1_{\mathcal{O}}=1_{\mathcal{O}}(x, t)$ are its associated characteristic functions, $\lambda, \alpha>0, y_{0} \in L^{2}(\Omega)$ and the potential $a=a(x, t) \in L^{\infty}(Q)$.

In this formulation, $\alpha$ plays the role of a penalty parameter which takes into account the approximate null controllability condition $\|y(T)\|_{L^{2}(\Omega)} \ll 1$, $\lambda$ is the amplitude of the bang-bang control, $\mathcal{O}$ depends on $(x, t)$ and represents the space-time region where the control takes its two values $+\lambda$ and $-\lambda$, and finally $q$ is the space-time region where the control is active.

This problem has been recently studied in [17] for the case in which $q=$ $\omega \times(0, T)$ is a cylinder, with $\omega \subset \subset \Omega$ a small subset, but in fact, with almost no changes, the results in [17] hold in this more general setting. Since the space $L^{\infty}(q ;\{0,1\})$ is not convex, the following relaxed problem was found in [17,
$\left(B_{\alpha}\right)\left\{\begin{array}{lll}\text { Minimize in }(\lambda, s): & J_{\alpha}(\lambda, s)=\frac{1}{2}\left(\lambda^{2}+\frac{1}{\alpha}\|y(T)\|_{L^{2}(\Omega)}^{2}\right) & \\ \text { subject to } & \\ & y_{t}-\Delta y+a y=[\lambda(2 s(x, t)-1)] 1_{q},(x, t) \in Q \\ & y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ & y(x, 0)=y_{0}(x) & x \in \Omega \\ & (\lambda, s) \in \mathbb{R}^{+} \times L^{\infty}(q ;[0,1]), & \end{array}\right.$
that is, the relaxation procedure simply consists of replacing $L^{\infty}(q ;\{0,1\})$ by its convex envelope.

Of course, the solution of $\left(B_{\alpha}\right)$ depends on the space-time region $q$ where the control acts. It is then natural to look for the best $q$ among those which have the same Lebesgue measure. In mathematical terms, we have the following time-dependent, nonlinear, optimal shape design problem:

$$
\left(P_{\alpha}\right) \begin{cases}\text { Minimize in } 1_{q}: & I_{\alpha}\left(1_{q}\right)=\frac{1}{2}\left(\lambda_{q}^{2}+\frac{1}{\alpha}\left\|y_{q}(T)\right\|_{L^{2}(\Omega)}^{2}\right) \\ \text { subject to } \\ & \left(\lambda_{q}, s_{q}\right) \text { is a solution of }\left(B_{\alpha}\right), \\ & |q|=L|Q|, \quad 0<L<1, \\ & 1_{q} \in L^{\infty}(Q ;\{0,1\}),\end{cases}
$$

where $y_{q}(x, t)$ is the solution of the heat equation associated to $\left(\lambda_{q}, s_{q}, 1_{q}\right)$ in problem $\left(B_{\alpha}\right)$ and $|\cdot|$ stands for the Lebesgue measure.

In one-space dimension, the case where the support of the control does not depend on the time variable and the null control is of minimal $L^{2}-$ norm was considered in [16]. In that case, there was a clear numerical evidence that the corresponding optimal design problem is ill-posed, that is, the solution is no longer a characteristic function but a density taking its values in the range $[0,1]$.

Our main goal in this work is twofold: (i) associate with $\left(P_{\alpha}\right)$ a well-posed relaxed problem. This is done in Section 2. And (ii), use this relaxation to solve numerically the original problem (Section 3).

We also would like to emphasize that the penalty approach for the approximate null controllability condition that we have considered from the very beginning highly simplifies the resolution of the problem. Indeed, for the case of the null control of minimal $L^{2}$ - norm, the relaxation method is based on a uniform, with respect to the location of the support, observability inequality for the solutions of the underlying adjoint problem. This kind of uniform observability inequality is, in our opinion, far from being easy to prove in higher dimensions. In fact, it is known only for the wave and heat equations in one space dimension [16,19,20]. The penalty approach avoids the need of using such an observability inequality. In addition, it lets include the time as a variable in the support, i.e., we are able to deal with supports that change with time.

The paper concludes with a short section of conclusions and related open problems.

## 2 Relaxation procedure

Consider the following two problems:
$\left(R B_{\alpha}\right)$

$$
\left\{\begin{array}{lll}
\text { Minimize in }(\lambda, s): & \bar{J}_{\alpha}(\lambda, s)=\frac{1}{2}\left(\lambda^{2}+\frac{1}{\alpha}\|y(T)\|_{L^{2}(\Omega)}^{2}\right) & \\
\text { subject to } & \\
& y_{t}-\Delta y+a y=[\lambda(2 s(x, t)-1)] \theta, & (x, t) \in Q \\
y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\
& y(x, 0)=y_{0}(x) & x \in \Omega \\
(\lambda, s) \in \mathbb{R}^{+} \times L^{\infty}(Q ;[0,1]), &
\end{array}\right.
$$

with $\theta=\theta(x, t) \in L^{\infty}(Q ;[0,1])$ satisfying the volume constraint

$$
\begin{equation*}
\int_{Q} \theta(x, t) d x d t=L|Q|, \quad 0<L<1 \tag{1}
\end{equation*}
$$

and

$$
\left(R P_{\alpha}\right)\left\{\begin{array}{l}
\text { Minimize in } \theta: \bar{I}_{\alpha}(\theta)=\frac{1}{2}\left(\lambda_{\theta}^{2}+\frac{1}{\alpha}\left\|y_{\theta}(T)\right\|_{L^{2}(\Omega)}^{2}\right) \\
\text { subject to } \\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\end{array} \lambda_{\theta}, s_{\theta}\right) \text { is a solution of }\left(R B_{\alpha}\right), ~(Q ;[0,1]) \text { and } \theta \text { satisfies (11). } .
$$

Our main result follows.
Theorem 1 Let us assume that $a=0$ or that $N=1$. Then $\left(R P_{\alpha}\right)$ is a true relaxation of $\left(P_{\alpha}\right)$ in the following sense:
(i) there exists at least one minimizer of $\left(R P_{\alpha}\right)$,
(ii) up to subsequences, every minimizing sequence, say $\left(1_{q_{n}}\right)$ of $\left(P_{\alpha}\right)$ converges to some $\theta \in L^{\infty}(Q ;[0,1])$ such that $\theta$ is a minimizer for $\left(R P_{\alpha}\right)$, and conversely,
(iii) if $\theta$ is a minimizer for $\left(R P_{\alpha}\right)$ and if $1_{q_{n}}$ converges to $\theta$ weak-* in $L^{\infty}(Q ;[0,1])$, then, up to a subsequence, $1_{q_{n}}$ is a minimizing sequence for $\left(P_{\alpha}\right)$.
To prove this result we shall need to invoke the first-order necessary optimality conditions of problem $\left(R B_{\alpha}\right)$.
Lemma 1 Let us assume that $a=0$ or that $N=1$ and that $\theta \in L^{\infty}(Q ;[0,1])$ satisfies (1). If $\left(\lambda^{\star}, s^{\star}\right)$ is a solution of $\left(R B_{\alpha}\right)$, then $s^{\star}$ is given by

$$
s^{\star}(x, t)=\left\{\begin{array}{l}
0 \text { if } p(x, t)<0 \text { and } \theta(x, t)>0  \tag{2}\\
1 \text { if } p(x, t)>0 \text { and } \theta(x, t)>0,
\end{array}\right.
$$

and $\lambda^{\star}=\|p \theta\|_{L^{1}(Q)}$, where $p=p(x, t) \in C\left([0, T] ; L^{2}(\Omega)\right) \cap L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$ solves the backward heat equation

$$
\left\{\begin{array}{l}
-p_{t}-\Delta p+a p=0, \quad(x, t) \in Q \\
p(\sigma, t)=0, \quad(\sigma, t) \in \Sigma, p(x, T)+\alpha^{-1} y(x, T)=0, x \in \Omega
\end{array}\right.
$$

$y=y(x, t)$ being the solution of the heat equation in problem $\left(R B_{\alpha}\right)$.

Proof. This result is essentially proved in 17] so that we only indicate the main steps. A straightforward computation shows that the functional $\bar{J}_{\alpha}$ is Gâteaux differentiable at each admissible $(\lambda, s)$ and its directional derivative in the admissible direction $(\widehat{\lambda}, \widehat{s})$ is given by

$$
\frac{\partial \bar{J}_{\alpha}(\lambda, s)}{\partial(\lambda, s)} \cdot(\widehat{\lambda}, \widehat{s})=\widehat{\lambda}\left(\lambda-\int_{Q} p(2 s-1) \theta d x d t\right)-2 \lambda \int_{Q} p \widehat{s} \theta d x d t .
$$

Consequently, if $\left(\lambda^{\star}, s^{\star}\right)$ is a solution of $\left(R B_{\alpha}\right)$, then

$$
\begin{align*}
\frac{\partial \bar{J}_{\alpha}\left(\lambda^{\star}, s^{\star}\right)}{\partial(\lambda, s)} \cdot\left(\lambda-\lambda^{\star}, s-s^{\star}\right) & =\left(\lambda-\lambda^{\star}\right)\left(\lambda^{\star}-\int_{Q} p\left(2 s^{\star}-1\right) \theta d x d t\right) \\
& -2 \lambda^{\star} \int_{Q} p\left(s-s^{\star}\right) \theta d x d t \tag{3}
\end{align*}
$$

for all $(\lambda, s) \in \mathbb{R}^{+} \times L^{\infty}(Q ;[0,1])$. Putting $\lambda=\lambda^{\star}$ in this expression and using an standard localization argument we get (2). Notice that at this point we are using the fact that in dimension $N=1$ or when the potential $a$ vanishes, the zero set of $p$ has zero Lebesgue measure (see [2]). Therefore, on the set $\{(x, t) \in Q: \theta(x, t)>0\}, s^{\star}$ is a characteristic function. Also, in the region where $\theta$ vanishes, the value of $s^{\star}$ is not of interest in our problem. Finally, if we put $s=s^{\star}$ in (3), then we obtain $\lambda^{\star}=\|p \theta\|_{L^{1}(Q)}$.

Proof of Theorem [1 Let us first prove that the functional $\bar{I}_{\alpha}(\theta)$ is sequentially continuous. Assume that $\theta_{n}, \theta \in L^{\infty}(Q ;[0,1])$ satisfy (1) and that

$$
\theta_{n} \rightharpoonup \theta \quad \text { weak }-\star \text { in } L^{\infty}(Q ;[0,1]) .
$$

Denote by $\left(\lambda_{n}, s_{n}\right)$ the solutions of $\left(R B_{\alpha}\right)$ associated to $\theta_{n}$.
Since $\left\{\lambda_{n}\right\}_{n \in \mathbb{N}}$ is (a part of) the solution of an optimization problem, it is bounded. Indeed, assume by contradiction that $\left\{\lambda_{n}\right\}_{n \in \mathbb{N}}$ is unbounded. Consider the constant sequence $\left(\bar{\lambda}_{n}=1, \bar{s}_{n} \equiv 1\right)$ associated to $\theta_{n}$ in problem $\left(R B_{\alpha}\right)$. Then, the solution $y^{n}$ of the system

$$
\begin{cases}y_{t}^{n}-\Delta y^{n}+a y^{n}=\left[\bar{\lambda}_{n}\left(2 \bar{s}_{n}(x, t)-1\right)\right] \theta_{n}, & (x, t) \in Q \\ y^{n}(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ y^{n}(x, 0)=y_{0}(x) & x \in \Omega\end{cases}
$$

satisfies

$$
\left\{\begin{array}{l}
y^{n} \rightharpoonup y \text { weakly in } L^{2}\left((0, T) ; H_{0}^{1}(\Omega)\right) \\
y_{t}^{n} \rightharpoonup y_{t} \text { weakly in } L^{2}\left((0, T) ; H^{-1}(\Omega)\right),
\end{array}\right.
$$

where $y=y(x, t)$ solves

$$
\begin{cases}y_{t}-\Delta y+a y=\theta, & (x, t) \in Q \\ y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ y(x, 0)=y_{0}(x) & x \in \Omega .\end{cases}
$$

By Aubin's lemma, up to a subsequence still labelled by $n$,

$$
y^{n} \rightarrow y \quad \text { strongly in } L^{2}\left((0, T) ; L^{2}(\Omega)\right)
$$

Hence, up to a subsequence,

$$
\begin{equation*}
y^{n}(t, \cdot) \rightarrow y(t, \cdot) \quad \text { strongly in } L^{2}(\Omega) \text { and a.e. } t \in[0, T] . \tag{4}
\end{equation*}
$$

Since $y^{n}(t)$ are continuous functions, convergence (4) in fact holds for all $t \in[0, T]$. In particular,

$$
\bar{J}_{\alpha}\left(\bar{\lambda}_{n}, \bar{s}_{n}\right)=\frac{1}{2}\left(1+\left\|y^{n}(T)\right\|_{L^{2}(\Omega)}^{2}\right) \rightarrow \frac{1}{2}\left(1+\|y(T)\|_{L^{2}(\Omega)}^{2}\right) \quad \text { as } n \rightarrow \infty
$$

Thus, for $n$ large enough, $\left(\bar{\lambda}_{n}, \bar{s}_{n}\right)$ is admissible for $\left(R B_{\alpha}\right)$ and gives a lower cost than $\left(\lambda_{n}, s_{n}\right)$, since we are assuming that $\left\{\lambda_{n}\right\}_{n \in \mathbb{N}}$ is unbounded. This contradicts the fact the $\left(\lambda_{n}, s_{n}\right)$ is a solution of $\left(R B_{\alpha}\right)$ associated to $\theta_{n}$.

Then, up to subsequences, still labelled by $n$, we have

$$
\left\{\begin{array}{l}
\lambda_{n} \rightarrow \lambda \\
s_{n} \rightharpoonup s \text { weak }-\star \text { in } L^{\infty}(Q ;[0,1]) .
\end{array}\right.
$$

Let us now prove that the convergence of the sequence $s_{n}$ to $s$ is in fact strong in $L^{2}$. Since $\lambda_{n}$ is bounded, by using the classical estimates for the heat equation, the solution of the system

$$
\begin{cases}y_{t}^{n}-\Delta y^{n}+a y^{n}=\left[\lambda_{n}\left(2 s_{n}(x, t)-1\right)\right] \theta_{n}, & (x, t) \in Q  \tag{5}\\ y^{n}(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ y^{n}(x, 0)=y_{0}(x) & x \in \Omega\end{cases}
$$

strongly converges in $L^{2}((0, T) \times \Omega)$ to $y$ solution of

$$
\begin{cases}y_{t}-\Delta y+a y=g, & (x, t) \in Q \\ y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ y(x, 0)=y_{0}(x) & x \in \Omega .\end{cases}
$$

with $g=g(x, t)$ the weak limit of $\left[\lambda_{n}\left(2 s_{n}(x, t)-1\right)\right] \theta_{n}$. As a consequence, the solution $p_{n}(x, t)$ of the adjoint system

$$
\left\{\begin{array}{l}
-\left(p_{n}\right)_{t}-\Delta p_{n}+a p_{n}=0,(x, t) \in Q \\
p_{n}(\sigma, t)=0, \quad(\sigma, t) \in \Sigma, p_{n}(x, T)+\alpha^{-1} y_{n}(x, T)=0, x \in \Omega,
\end{array}\right.
$$

also converges strongly in $L^{2}((0, T) \times \Omega)$ to some $p$. In particular, the sequence

$$
p_{n}^{+}(x, t)= \begin{cases}p_{n}(x, t) & \text { if } p_{n}(x, t)>0 \\ 0 & \text { if } p_{n}(x, t) \leq 0,\end{cases}
$$

strongly converges to the positive part of $p$, that we denote by $p^{+}$. Moreover, since $s_{n}$ satisfies (2),

$$
p_{n} s_{n}=p_{n}^{+} \rightarrow p^{+}=p s \quad \text { strongly in } L^{2}(Q) .
$$

Replacing $\lambda^{\star}=\lambda_{n}, \lambda=\lambda_{n}, s^{\star}=s_{n}, s=\bar{s} \in L^{\infty}(Q ;[0,1])$ and $\theta=\theta_{n}$ in (3) we may pass to the limit in (3) to conclude that

$$
\int_{Q} p(\bar{s}-s) \theta d x d t \leq 0
$$

for all $\bar{s} \in L^{\infty}(Q ;[0,1])$. Notice that $\lambda$ (limit of $\left.\lambda_{n}\right)$ can not vanish. Reasoning as in the proof of Lemma 1 we have that $s$ is a characteristic function. Thus, since both $s_{n}$ and $s$ are characteristic functions,

$$
\begin{equation*}
s_{n} \rightarrow s \text { strongly in } L^{p}(Q) \text { for all } 1 \leq p<\infty \tag{6}
\end{equation*}
$$

We refer to [1, Remark 3.3] for details on this passage. Hence,

$$
\left[\lambda_{n}\left(2 s_{n}(x, t)-1\right)\right] \theta_{n} \rightharpoonup[\lambda(2 s(x, t)-1)] \theta \quad \text { in the sense of distributions. }
$$

Reasoning as above, up to subsequences, the solution $y^{n}$ of the system (5) satisfies

$$
\begin{equation*}
y^{n}(t, \cdot) \rightarrow y(t, \cdot) \quad \text { strongly in } L^{2}(\Omega) \text { and a.e. } t \in[0, T] \tag{7}
\end{equation*}
$$

where $y=y(x, t)$ solves

$$
\begin{cases}y_{t}-\Delta y+a y=[\lambda(2 s(x, t)-1)] \theta, & (x, t) \in Q \\ y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ y(x, 0)=y_{0}(x) & x \in \Omega .\end{cases}
$$

As before, since $y^{n}(t)$ are continuous functions, convergence (7) in fact holds for all $t \in[0, T]$. In particular,

$$
\begin{equation*}
\bar{I}_{\alpha}\left(\theta_{n}\right)=\frac{1}{2}\left(\lambda_{n}^{2}+\left\|y^{n}(T)\right\|_{L^{2}(\Omega)}^{2}\right) \rightarrow \frac{1}{2}\left(\lambda^{2}+\|y(T)\|_{L^{2}(\Omega)}^{2}\right) \quad \text { as } n \rightarrow \infty . \tag{8}
\end{equation*}
$$

Let us now prove that $(\lambda, s)$ is a solution to $\left(R B_{\alpha}\right)$ associated with $\theta$. Assume, by contradiction, that there exists $(\bar{\lambda}, \bar{s})$, admissible for $\left(R B_{\alpha}\right)$ and associated to the same $\theta$, such that

$$
\begin{equation*}
\frac{1}{2}\left(\bar{\lambda}^{2}+\|\bar{y}(T)\|_{L^{2}(\Omega)}^{2}\right)<\frac{1}{2}\left(\lambda^{2}+\|y(T)\|_{L^{2}(\Omega)}^{2}\right) \tag{9}
\end{equation*}
$$

Now, we look at $(\bar{\lambda}, \bar{s})$ as an admissible state for $\left(R B_{\alpha}\right)$ associated with the sequence $\theta_{n}$. Reasoning as before, for each $\varepsilon>0$ there exists $n_{0} \in \mathbb{N}$ such that

$$
\bar{I}_{\alpha}\left(\theta_{n}\right)<\frac{1}{2}\left(\bar{\lambda}^{2}+\|\bar{y}(T)\|_{L^{2}(\Omega)}^{2}\right)+\varepsilon \quad \forall n \geq n_{0} .
$$

Passing to the limit in this expression and taking into account (8) we have

$$
\begin{equation*}
\frac{1}{2}\left(\lambda^{2}+\|y(T)\|_{L^{2}(\Omega)}^{2}\right) \leq \frac{1}{2}\left(\bar{\lambda}^{2}+\|\bar{y}(T)\|_{L^{2}(\Omega)}^{2}\right)+\varepsilon \tag{10}
\end{equation*}
$$

Since $\varepsilon$ is arbitrary, (10) is a contradiction with (9). Thus,

$$
\bar{I}_{\alpha}(\theta)=\frac{1}{2}\left(\lambda^{2}+\|y(T)\|_{L^{2}(\Omega)}^{2}\right)
$$

and therefore (8) shows that $\bar{I}_{\alpha}$ iscontinuous.
This, together with the compactness of the weak- topology, proves that $\left(R P_{\alpha}\right)$ has a solution.

Statements (ii) and (iii) are a straightforward consequence of the continuity of $\bar{I}_{\alpha}$ and of the fact that the closedness, with respect to the weak- $\star$ topology of $L^{\infty}$, of the set of characteristic functions $1_{q} \in L^{\infty}(Q ;\{0,1\})$ having a fixed Lebesgue measure $|q|=L|Q|$ is equal to the space of densities $\theta \in$ $L^{\infty}(Q ;[0,1])$ which satisfy (1). We refer to [12, Proposition 7.2.14] for more details on this last passage.

## 3 Numerical simulations

In this section we address the numerical resolution of problem $\left(R P_{\alpha}\right)$. We first describe the algorithm of minimization and then show some numerical experiments. Before this, we obtain the following equivalent form of problem $\left(R P_{\alpha}\right)$.

Consider the new problem


Then, we have:
Proposition 1 Problems $\left(R P_{\alpha}\right)$ and $\left(\overline{R P}_{\alpha}\right)$ are equivalent in the following sense: if $\theta$ is a solution of $\left(R P_{\alpha}\right)$ and $\left(\lambda_{\theta}, s_{\theta}\right)$ is a solution of $\left(R B_{\alpha}\right)$ associated to $\theta$, then $\left(\lambda_{\theta}, s_{\theta}, \theta\right)$ is a solution of $\left(\overline{R P}_{\alpha}\right)$, and conversely, if $(\lambda, s, \theta)$ is a solution of $\left(\overline{R P}_{\alpha}\right)$, then $\theta$ solves $\left(R P_{\alpha}\right)$ and $(\lambda, s)$ is a solution of $\left(R B_{\alpha}\right)$ associated to $\theta$.

Proof. Let $\theta$ be a solution of $\left(R P_{\alpha}\right)$ and denote by $\left(\lambda_{\theta}, s_{\theta}\right)$ the corresponding solution of $\left(R B_{\alpha}\right)$. We argue by contradiction and assume that there exists $(\bar{\lambda}, \bar{s}, \bar{\theta})$, admissible for $\left(\overline{R P}_{\alpha}\right)$, such that

$$
\begin{equation*}
\frac{1}{2}\left(\bar{\lambda}^{2}+\frac{1}{\alpha}\|\bar{y}(T)\|_{L^{2}(\Omega)}^{2}\right)<\frac{1}{2}\left(\lambda_{\theta}^{2}+\frac{1}{\alpha}\left\|y_{\theta}(T)\right\|_{L^{2}(\Omega)}^{2}\right) . \tag{11}
\end{equation*}
$$

Since $\bar{\theta}$ is admissible for $\left(R P_{\alpha}\right)$ and $\theta$ is a solution of $\left(R P_{\alpha}\right)$,

$$
\begin{equation*}
\bar{I}_{\alpha}(\theta) \leq \bar{I}_{\alpha}(\bar{\theta}) . \tag{12}
\end{equation*}
$$

Denote by $\left(\lambda_{\bar{\theta}}, s_{\bar{\theta}}\right)$ the corresponding solution of $\left(R B_{\alpha}\right)$ associated to $\bar{\theta}$. Since $(\bar{\lambda}, \bar{s})$ is admissible for $\left(R B_{\alpha}\right)$,

$$
\frac{1}{2}\left(\lambda \frac{2}{\theta}+\frac{1}{\alpha}\left\|y_{\bar{\theta}}(T)\right\|_{L^{2}(\Omega)}^{2}\right) \leq \frac{1}{2}\left(\bar{\lambda}^{2}+\frac{1}{\alpha}\|\bar{y}(T)\|_{L^{2}(\Omega)}^{2}\right) .
$$

But, from (12) it follows that

$$
\begin{align*}
\frac{1}{2}\left(\lambda_{\theta}^{2}+\frac{1}{\alpha}\left\|y_{\theta}(T)\right\|_{L^{2}(\Omega)}^{2}\right) & \leq \frac{1}{2}\left(\lambda_{\theta}^{2}+\frac{1}{\alpha}\left\|y_{\bar{\theta}}(T)\right\|_{L^{2}(\Omega)}^{2}\right)  \tag{13}\\
& \leq \frac{1}{2}\left(\bar{\lambda}^{2}+\frac{1}{\alpha}\|\bar{y}(T)\|_{L^{2}(\Omega)}^{2}\right)
\end{align*}
$$

which is a contradiction with (11).
Conversely, let $(\lambda, s, \theta)$ be a solution of $\left(\overline{R P}_{\alpha}\right)$. As before, we argue by contradiction and assume that there exists $\bar{\theta}$, admissible for $\left(R P_{\alpha}\right)$ such that

$$
\begin{equation*}
\frac{1}{2}\left(\lambda \frac{2}{\theta}+\frac{1}{\alpha}\left\|y_{\bar{\theta}}(T)\right\|_{L^{2}(\Omega)}^{2}\right)<\frac{1}{2}\left(\lambda^{2}+\frac{1}{\alpha}\|y(T)\|_{L^{2}(\Omega)}^{2}\right) \tag{14}
\end{equation*}
$$

where $\left(\lambda_{\bar{\theta}}, s_{\bar{\theta}}\right)$ is a solution of $\left(R B_{\alpha}\right)$ associated to $\bar{\theta}$. Since $\left(\lambda_{\bar{\theta}}, s_{\bar{\theta}}, \bar{\theta}\right)$ is admissible for $\left(\overline{R P}_{\alpha}\right)$, (14) implies that $(\lambda, s, \theta)$ is not a solution to $\left(\overline{R P}_{\alpha}\right)$, which is a contradiction. This completes the proof.

At the numerical level, problem $\left(\overline{R P}_{\alpha}\right)$ is simpler to solve than $\left(R P_{\alpha}\right)$. Thus, from now on we focus on the numerical resolution of $\left(\overline{R P}_{\alpha}\right)$.

### 3.1 Algorithm of minimization

### 3.1.1 Numerical resolution of problem $\left(\overline{R P}_{\alpha}\right)$

As usual, the volume constraint on $\theta$ is incorporated in the cost function through a Lagrange multiplier. Therefore, we consider the augmented cost

$$
\begin{equation*}
\bar{I}_{\alpha}^{\star}(\lambda, s, \theta)=I_{\alpha}^{\star}(\lambda, s, \theta)+\gamma\left(\int_{Q} \theta d x d t-L|Q|\right) . \tag{15}
\end{equation*}
$$

A simple computation shows that the gradient of $\bar{I}_{\alpha}^{\star}$ at the point $(\lambda, s, \theta)$ is given by

$$
\begin{equation*}
\nabla \bar{I}_{\alpha}^{\star}(\lambda, s, \theta)=\left(\lambda-\int_{Q} p(2 s-1) \theta d x d t,-2 \lambda p \theta,-\lambda(2 s-1) p+\gamma\right) \tag{16}
\end{equation*}
$$

where the adjoint state $p$ solves the backward equation

$$
\begin{cases}-p_{t}-\Delta p+a p=0 & \text { in } Q  \tag{17}\\ p=0 & \text { on } \Sigma \\ p(T)=-\alpha^{-1} y(T) & \text { in } \Omega .\end{cases}
$$

We propose the following descent algorithm to solve $\left(\overline{R P}_{\alpha}\right)$ :

1. Initialization: take $\left(\lambda^{0}, s^{0}, \theta^{0}\right) \in \mathbb{R}^{+} \times L^{\infty}(Q ;[0,1]) \times L^{\infty}(Q ;[0,1])$, with $\left\|\theta^{0}\right\|_{L^{1}(Q)}=L|Q|$.
2. For $k \geq 0$, iteration until convergence as follows:

- Computation of the solution $u_{\left(\lambda^{k}, s^{k}, \theta^{k}\right)}$ of the state law in problem $\left(\overline{R P}_{\alpha}\right)$ and then the solution $p_{u_{\left(\lambda^{\left.k, s^{k}, \theta^{k}\right)}\right.}}$ of the adjoint system (17).
- Computation of the descent direction $-\nabla \bar{I}_{\alpha}^{\star}\left(\lambda^{k}, s^{k}, \theta^{k}\right)$ as given by (16) where the corresponding multiplier $\gamma^{k}$ is chosen in such a way that $\left\|\theta^{k+1}\right\|_{L^{1}(Q)}=L|Q|$.
- Update the optimization variables, namely,

$$
\left\{\begin{array}{l}
\lambda^{k+1}=\lambda^{k}-\varepsilon_{1}^{k}\left(\lambda^{k}-\int_{Q} p^{k}\left(2 s^{k}-1\right) \theta^{k} d x d t\right) \\
s^{k+1}=P_{[0,1]}\left(s^{k}+2 \varepsilon_{2}^{k} \lambda^{k} p^{k} \theta^{k}\right) \\
\theta^{k+1}=\theta^{k}-\varepsilon_{3}^{k}\left(-\lambda^{k}\left(2 s^{k}-1\right) p^{k}+\gamma^{k}\right)
\end{array}\right.
$$

where $P_{[0,1]}(x)=\max (0, \min (1, x))$ is the projection of $x$ on $[0,1]$. The positive step-size parameters $\varepsilon_{1}^{k}, \varepsilon_{2}^{k}$ and $\varepsilon_{3}^{k}$ are chosen small enough as to ensure that $\lambda^{k+1}>0, \theta^{k+1} \in L^{\infty}(Q ;[0,1])$ and, in addition, there is a decrease in the cost function. Notice that we use three different parameters for $\lambda$ and the densities $s$ and $\theta$.

As for the stopping criterium we take

$$
\begin{equation*}
\left|\bar{I}_{\alpha}^{\star}\left(\lambda^{n+1}, s^{n+1}, \theta^{n+1}\right)-\bar{I}_{\alpha}^{\star}\left(\lambda^{n}, s^{n}, \theta^{n}\right)\right| \leq \operatorname{tol} \bar{I}_{\alpha}^{\star}\left(\lambda^{0}, s^{0}, \theta^{0}\right) \tag{18}
\end{equation*}
$$

with a tolerance $t o l=10^{-5}$ in the experiments that follow.

### 3.1.2 From an optimal relaxed density $\theta$ to a quasi-optimal domain $q_{M}$.

Theorem 1, part (iii), may be used to construct a minimizing sequence for the original problem $\left(P_{\alpha}\right)$. Indeed, once a solution $\theta$ of problem $\left(\overline{R P}_{\alpha}\right)$ has been computed by using the algorithm just described, we proceed as follows.

For simplicity, let us assume that $\Omega=] 0,1\left[{ }^{2}\right.$ is the unit square. To begin with, we make a regular partition of $\Omega$ in a number of sub-rectangles $\Omega_{n}$, with $1 \leq n \leq M$, for some fixed $M \in \mathbb{N}$. Then, we compute the integral of $\theta$ over each one of such sub-rectangles at time $t$. Let us denote by $m_{n}^{t}$ the value of these integrals. According to these values, we then define the squares $q_{n}^{t}$ centred at the center of $\Omega_{n}$ and having an area equal to $m_{n}^{t}$. Finally, the space-time domain $q_{M}$ is defined as the union of $q_{n}^{t}$. This procedure preserves the volume constraint (1) and, as simulation results in next section show, provides satisfactory results.

### 3.2 Numerical experiments

Next, we present two numerical experiments in 2D to illustrate the theoretical results of the preceding section. We consider both a regular and a discontinuous initial datum. Our main objectives in this section are:
(a) analyze numerically the well-ill posedness character of problem $\left(P_{\alpha}\right)$,
(b) in the case in which $\left(P_{\alpha}\right)$ is ill-posed, compute a quasi-optimal timedependent domain following the approach described in Subsection 3.1.2
(c) test numerically the influence of the time-dependent character of the shape and position of the region where the control is active, and
(d) analyze numerically the effect of small perturbations in the initial datum on the optimal relaxed density.

In order to have a better control of the heat diffusion during the time interval, throughout this section we replace the operator $\partial_{t}-\Delta$ by $\partial_{t}-c \Delta$, with $c$ lower than 1 .

The algorithm described in the preceding section has been implemented in the free software FreeFem++-cs 3.19-1 (http://www.freefem.org/).

## Experiment 1: smooth initial datum

We take $\Omega=] 0,1\left[{ }^{2}\right.$ the unit square, $y_{0}\left(x_{1}, x_{2}\right)=\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right), c=0.1$, $T=0.5$, the potential $a=0$ and the penalty parameter $\alpha=10^{-4}$. The state and adjoint state equations have been solved by using a backward Euler scheme in time with $\Delta t=0.05$ and $P_{1}$ finite elements for the spatial variable on a uniform triangular mesh. Precisely, the unit square $\Omega=] 0,1\left[{ }^{2}\right.$ has been decomposed into $100 \times 100$ non-overlapping sub-rectangles, which we split up into two equal triangles. The algorithm has been initialized with the following parameters: $\lambda^{0}=2, s^{0}(x, t)=0.2$ on $Q$ and $\theta^{0}(x, t)=0.25$ on $Q$, with $x=\left(x_{1}, x_{2}\right)$. As for the volume constraint we take $L=0.25$.

After convergence of the algorithm (755 iterations) we obtain $s(x, t)=0$, except near the boundary where $s$ takes some values in $] 0,1[$. This is due to the
boundary condition that the adjoint state $p$ satisfies. Notice that $s$ is updated through $p$ and hence it is difficult to move $s$ from its inital value near the boundary. Fortunately, this numerical error is compensated by the density $\theta$ which is almost zero close to the boundary. Anyway, no oscillations in the control is observed for this initial condition. This is in agreement with the fact that the initial condition $y_{0}$ is non-negative. As usual, an exponential decrease in the cost function is obtained.

Table 1 collects results for the solution of problem $\left(\overline{R P}_{\alpha}\right)$ (first column) and results obtained from the quasi-optimal domain $q_{M}$, with $M=25$, as described in Subsection 3.1.2 (second column). We notice that the optimal $s$ and $\lambda$ for the optimal density $\theta$ are not optimal for $q_{M}$. For this reason, after computing $q_{M}$ we perform a few more iterations to obtain new $s$ and $\lambda$ which are closer to the optimal ones for $q_{M}$. In particular, as in the case where the domain does not depend on time, the density $s$ oscillates near the final time and takes some values in $(0,1)$. A very fine mesh together with a large number of iterates is needed to recover a bi-valued function. We do not enter in details on this issue here because it has been deeply studied in [17.

Figure 1 shows the pictures of the optimal density $\theta(x, t)$ in $\Omega$ and of its associated quasi-optimal time-dependent domain $q_{M}$ at the times $t=$ $0.1,0.3,0.5$. We observe that the optimal $\theta(x, t)$ is not a characteristic function. This seems to indicate that a true relaxation phenomenon occurs even for a very regular initial condition. A similar qualitative result was observed for the $L^{2}$-case in one space dimension and time-independent domains (see [16]). Also, it is observed the strong dependence of $\theta(x, t)$ with respect to the time variable and therefore, the quasi-optimal domain $q_{M}$ is not a cylinder as it also depends on time. Notice that for every $x \in \Omega$ the function $t \mapsto \theta(x, t)$ is increasing. For instance, the values of $\theta(0.5,0.5 ; t)$ range from 0.35098 for $t=0.1$ (Figure 1 top left) to 0.90933 , for $t=0.5$ (Figure $\mathbb{1}$ bottom left). Consequently, the control system takes advantage of the natural dissipation of the heat equation and thus concentrates the action of the control mainly near the final time $T$.

Finally, to test numerically the dependence of the optimal relaxed density $\theta$ with respect to the initial condition, $y_{0}(x)$ has been affected by additive gaussian white noise with several standard deviations $\sigma$. More precisely, to generate a normal distribution function we have used the Box-Muller transformation

$$
\overline{y_{0}}=\sqrt{-2 \log \left(X_{1}\right) \cos \left(2 \pi X_{2}\right)}
$$

where both $X_{1}$ and $X_{2}$ are independent random variables that are uniformly distributed in the interval $(0,1]$. Thus, we have perturbed point-wise the initial condition to obtain $y(x, 0)=y_{0}(x)\left(1+\sigma \overline{y_{0}}\right)$. With this new initial datum and for several values of the level noise $\sigma$ we have solved problem $\left(\overline{R P}_{\alpha}\right)$ to obtain a new optimal density $\theta_{\sigma}(x, t)$. Table 2 displays results of $\left\|\theta-\theta_{\sigma}\right\|_{L^{1}(Q)}$ and

Table 1 - Experiment 1- First column displays results for the optimal density $\theta(x, t)$, solution of $\left(\overline{R P}_{\alpha}\right)$. Second column shows results of its associated quasi-optimal time-dependent domain $q_{25}$.

|  | optimal relaxed density $\theta$ | quasi-optimal domain $q_{M}$ |
| :--- | :--- | :--- |
| $\\|y(T)\\|_{L^{2}(\Omega)}$ | $1.51644 \times 10^{-3}$ | $5.5333 \times 10^{-3}$ |
| $\lambda$ | 1.91110 | 2.00497 |

Table 2 - Experiment 1- Values of $\left\|\theta-\theta_{\sigma}\right\|_{L^{1}(Q)}$ and $\left|\lambda-\lambda_{\sigma}\right|$ for $\sigma=0.1$ and $\sigma=0.01$.

|  | $\sigma=0.1$ | $\sigma=0.01$ |
| :--- | :--- | :--- |
| $\left\\|\theta-\theta_{\sigma}\right\\|_{L^{1}(Q)}$ | $6.1326 \times 10^{-3}$ | $6.09936 \times 10^{-3}$ |
| $\left\|\lambda-\lambda_{\sigma}\right\|$ | 0.0009 | 0.000167 |

$\left|\lambda-\lambda_{\sigma}\right|$ for $\sigma=0.1$ and $\sigma=0.01$. A very low variation of $\theta$, solution of $\left(\overline{R P}_{\alpha}\right)$, with respect to the initial datum $y_{0}$ is observed.

## Experiment 2: discontinuous initial datum

As in the preceding experiment, we take $\Omega=] 0,1\left[{ }^{2}\right.$ the unit square, $c=0.1$ and $T=0.5$. As for the penalty parameter, now $\alpha=10^{-5}$. We consider the constant potential $a=-1$ over $Q$ and the discontinuous initial condition $y_{0}=-1_{] 0.1,0.4[\times] 0.1,0.4[ }+1_{] 0.6,0.9[\times] 0.6,0.9[ }$. For the volume constraint we take $L=0.1613658$. The state and adjoint state equations have been solved as in the preceding experiment.

As for the initialization of the algorithm, $\lambda^{0}=1.5$ and, in order to favour a quick convergence of the densities $s$ and $\theta$, we take $s^{0}\left(x_{1}, x_{2}, t\right)=1_{\left\{\left(x_{1}+x_{2}<1\right\}\right.}$ and
$\theta^{0}\left(x_{1}, x_{2}, t\right)=0.5 e^{2 t-1}\left[e^{-9\left(\left(x_{1}-0.25\right)^{2}+\left(x_{2}-0.25\right)^{2}\right)}+e^{-9\left(\left(x_{1}-0.75\right)^{2}+\left(x_{2}-0.75\right)^{2}\right)}\right]$.
Indeed, in this case the stopping criterium (18) is satisfied after 135 iterates.
Table 3 collects results for the solution of problem $\left(\overline{R P}_{\alpha}\right)$ (first column) and results obtained from the quasi-optimal domain $q_{M}$, with $M=25$, as described in Subsection 3.1.2 (second column).

The optimal density $\theta(x, t)$ is also a non-characteristic function which concentrates mainly in the support of the initial datum $y_{0}$ and shows a variation with respect to the time variable. As for the density $s(x, t)$, up to numerical approximation errors, it is also a bi-valued function which, as expected, takes the value 1 in the region $x_{1}+x_{2} \leq 1$ and zero in the rest. As an illustration of the results, Figure 2 displays the iso-values of $\theta(x, 0.5)$ (left) and $s(x, 0.5)$ (right).


Fig. 1 - Experiment 1- Left column displays results for the optimal density $\theta(x, t)$ in $\Omega$ for $t=0.1$ (top) $t=0.3$ (middle), and $t=0.5$ (bottom). Right column shows results of its associated quasi-optimal time-dependent domain $q_{25}$ for the same discrete times $t=$ $0.1,0.3,0.5$. The region where the control is active and is equal to $-\lambda$ is in blue color. Red color is for the region where the control is not active.

Table 3 - Experiment 2- First column displays results for the optimal density $\theta(x, t)$, solution of $\left(\overline{R P}_{\alpha}\right)$. Second column shows results of its associated quasi-optimal time-dependent domain $q_{25}$.

|  | optimal relaxed density $\theta$ | quasi-optimal domain $q_{M}$ |
| :--- | :--- | :--- |
| $\\|y(T)\\|_{L^{2}(\Omega)}$ | $5.78899 \times 10^{-4}$ | $1.6125 \times 10^{-3}$ |
| $\lambda$ | 1.08674 | 1.29975 |

Table 4 - Experiment 2- Values of $\left\|\theta-\theta_{\sigma}\right\|_{L^{1}(Q)}$ and $\left|\lambda-\lambda_{\sigma}\right|$ for $\sigma=0.1$ and $\sigma=0.01$.

|  | $\sigma=0.1$ | $\sigma=0.01$ |
| :--- | :--- | :--- |
| $\left\\|\theta-\theta_{\sigma}\right\\|_{L^{1}(Q)}$ | $3.95225 \times 10^{-3}$ | $3.82754 \times 10^{-3}$ |
| $\left\|\lambda-\lambda_{\sigma}\right\|$ | 0.00621 | 0.00062 |

Figure 3 shows the quasi-optimal control $\lambda(2 s-1) 1_{q_{25}}$ at times $t=0$, $t=0.2$ and $t=0.4$ (right column) and its associated controlled solution $y(x, t)$ at the times $t=0, t=0.3$ and $t=0.5$ (right column). As in the preceding experiment, we observe that the density $s$ associated with $q_{25}$ (contrary to what happens with the density associated to $\theta$ ) oscillates near the final time. We refer again to [17] for more details on this phenomenon.


Fig. 2 - Experiment 2- Pictures of the optimal densities $\theta(x, 0.5)$ (left) and $s(x, 0.5)$ (right).

Similarly to Experiment 1, Table 4 collects results of $\left\|\theta-\theta_{\sigma}\right\|_{L^{1}(Q)}$ and $\left|\lambda-\lambda_{\sigma}\right|$ for $\sigma=0.1$ and $\sigma=0.01$.

## 4 Conclusions and related open problems

In this paper, we have considered the problem of determining the best shape and position of the support of optimal controls in the $L^{\infty}$-norm (a fortiori, bang-bang type controls) for the approximate controllability of a linear heat


Fig. 3 - Experiment 2- Left column displays results for the controlled solution $y(x, t)$ at times $t=0$ (top), $t=0.2$ (middle) and $t=0.5$ (bottom). Right column shows results for its associated quasi-optimal control $\lambda(2 s-1) 1_{q_{25}}$ at times $t=0$ (top), $t=0.2$ (middle) and $t=0.4$ (bottom). Control is inactive in the grey color region, control is equal to $+\lambda$ in the black region, and to $-\lambda$ in the white region.
equation with a bounded potential. One of the main novelties with respect to previous related works is that the shape and position of the support of the control depends on the time variable. Since the problem is not convex, a well-posed relaxed formulation has been obtained and a numerical algorithm for the resolution of the relaxed problem has been proposed and tested in two numerical experiments.

Numerical simulation results seem to indicate that, even for very regular initial conditions, the original problem is ill-posed and therefore a true relax-
ation phenomenon occurs in this context. Anyway, a quasi-optimal solution of the original problem is obtained from the solution of the relaxed one.

Although the initial condition of the state law is fixed from the very beginning, the dependence of the solution of the design problem with respect to small perturbations of the initial datum has been numerically analyzed. Therefore, the approach of this paper provides some insights to the problem of optimal design of support of the control not only for a single initial datum but for a set of initial data with support localized in an specific region.

To conclude, let us mention the following related open problems:

- The results of this paper may be extended to the case where the control acts on a part of the boundary and the state law is a more general linear parabolic equation or even a semilinar parabolic equation. For the case in which the support of the control does not depend on time, some positive controllability results in the linear and semilinear case exists both at the theoretical level in the case of optimal controls in the $L^{\infty}$ _norm [6 and at the numerical one in the $L^{2}$-norm [3,7.
- Also, the case where the support of the control has the form

$$
q=\omega_{t} \times(0, T), \quad \text { with }\left|\omega_{t}\right|=L|\Omega| \quad \forall t
$$

i.e., the volume constraint is satisfied at each time slice, can be studied in the same manner. In particular Theorem 1 and the numerical algorithm proposed here hold in this case. The only difference is that the Lagrange multiplier in equation (15) should depend on time.

- Consider the approximate controllability problem for the heat equation where the control acts on a curve $\gamma:[0, T] \rightarrow \Omega$. Here, the goal is to compute the optimal control $f=f(x, t)$ in $L^{2}$-norm (or in $L^{\infty}$-norm) such that for a given $\epsilon \ll 1$ the solution of the problem

$$
\begin{cases}y_{t}-\Delta y=f(x, t) \delta_{\gamma(t)}(x), & (x, t) \in Q \\ y(\sigma, t)=0, & (\sigma, t) \in \Sigma \\ y(x, 0)=y_{0}(x) & x \in \Omega .\end{cases}
$$

satisfies $\|y(T)\|_{L^{2}(\Omega)} \leq \epsilon$. For the case in which the curve $\gamma$ is fixed, this problem has been studied in [5] and also in [4 in the case of the wave equation. A natural question that arises is the optimal design of the curve $\gamma$, i.e., the computation of the curve that best minimizes the norm of the control. This situation may be considered as a limit case (when the parameter for the volume constraint $L \rightarrow 0$ ) in a similar formulation as in the present paper.

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