Numerical simulation of the boundary exact control for the system of linear elasticity☆

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Abstract

The problem of computing numerically the boundary exact control for the system of linear elasticity in 2D is addressed. A numerical method which has been recently proposed in [Stud. Appl. Math. 121 (2008), no. 1, 27–47] is implemented. Two cases are considered: first, a rectangular domain with Dirichlet controls acting on two adjacent edges, and secondly, a circular domain with Neumann controls distributed along the whole boundary.

Key words: boundary exact controllability, system of linear elasticity, numerical simulation.
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1. Introduction - Problem Formulation

During the last decades important progress has been made in the exact controllability of distributed parameter systems both from a theoretical and numerical point of view. After the works by D. L. Russell [8], J. L. Lions [5] and many others, the mathematical linear theory is very well established. Regarding the numerical resolution of controllability problems, a lot of work has been also carried out since the pioneering works by R. Glowinski et al. (see [4] and the references there in). But, even so, the development of numerical methods for solving some of these problems is still a challenge. An important difficulty arises in the fact that numerical schemes that are stable for solving simple initial-boundary value problems (like the 1D wave equation) may be unstable in exact controllability [10]. Thus, the method may fail which consists in (a) approximating distributed-parameter control systems by finite-dimensional control systems, (b) computing the family of controls of such a systems and (c) recovering the control of the original system as the limit (when the size mesh goes to zero) of the finite-dimensional controls.

In this note, we consider the problem of computing numerically the boundary exact control for the free vibrations of a two-dimensional homogeneous and isotropic elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$. For the case of Dirichlet-type controls acting on a part, say Γ_1, of Γ, and given initial data $\left(u^0(x), u^1(x)\right)$ in a suitable function space, the problem of exact controllability for the system of linear elasticity refers to the existence of a positive time T and a control function $v = v(x,t)$ such that the solution $u = u(x,t)$ of the system

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = 0 & \text{in} \quad Q = \Omega \times (0,T) \\ u = 0 & \text{on} \quad \sum_0 = \Gamma_0 \times (0,T) \\ u = v & \text{on} \quad \sum_1 = \Gamma_1 \times (0,T) \\ (u(0), u_t(0)) = \left(u^0, u^1\right) & \text{in} \quad \Omega \end{cases} \tag{1}$$

satisfies the null controllability condition

$$u(T) = u_t(T) = 0 \quad \text{in } \Omega. \tag{2}$$

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As usual, t stands for the time variable, $x=(x_1,x_2)$ is the spatial variable, $u(x,t)=(u_1(x,t),u_2(x,t))$ is the displacement of the material point x at time t, $\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$ is the Laplacian, $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)$ is the gradient, the boundary $\Gamma=\Gamma_0\cup\Gamma_1$ is composed of two disjoint parts, and $\lambda,\mu>0$ are Lamé's coefficients.

From a theoretical point of view, this problem has already been analyzed and solved. We refer for instance to [1] and to [5, Ch. IV] for some positive results concerning the existence of solutions in the usual function spaces. We also notice that due to the finite velocity propagation of elastic waves, the controllability condition (2) cannot hold for arbitrary small time T. Therefore, there is a minimal time, say $T^\star>0$, which depends on Ω, Γ_0 and on Lamé coefficients, for which problem (1)-(2) has a solution. Since this work is mainly devoted to numerical simulation we do not enter here in these (important) details and refer the reader to the above mentioned references [1, 5].

The aim of this note is to implement a numerical method for the resolution of problem (1)-(2). Our approach is based on Russell's ideas [7, 8]. The convergence of the numerical algorithm follows easily from the fact that the elasticity system in $\mathbb{R}^2$ locally dissipates its energy [6]. As we will see later on, the main advantage of this method is that it applies to general geometries and boundary conditions. In addition, it requires very simple mathematical tools. Indeed, only the use of the Fast Fourier Transform (FFT) for solving some associated Cauchy problems is needed. As a consequence, this method does not generate spurious high frequency solution components.

The rest of the paper is organized as follows. In Section 2, we briefly describe the numerical method employed for computing the boundary controls. In Section 3, the algorithm is implemented in two cases. First, we solve system (1)-(2) with Ω the unit square and Γ_1 two adjacent edges. The control is of Dirichlet-type, that is, the control function is the displacement field on a part of the boundary. Secondly, we consider a circular domain and the controls are distributed along the whole boundary in the form of a density of forces (Neumann-type control).

2. Description of the numerical scheme

In this section, we briefly describe the numerical algorithm for solving the controllability problem (1)-(2). For a detailed analysis of this method (including convergence and computational cost of its computer implementation) we refer the reader to [6].

To fix ideas, let us assume that $\Omega\equiv R_1=(0,1)^2$ is the unit square, Γ_0 is composed of the two edges $x_1=0$ and $x_2=0$, and Γ_1 is the rest of the boundary, i.e.,

$$\Gamma_1=\left\{(1,s)\in\mathbb{R}^2:\quad 0<s\leq 1,\ \right\}\cup\left\{(s,1)\in\mathbb{R}^2:\quad 0<s\leq 1\right\}.$$

The algorithm is structured as follows:

Step 1. We begin by extending the initial data $\left(u^0,u^1\right)$ of system (1) to all of $\mathbb{R}^2$. To this end, consider the rectangles

$$R_2=(-1,0)\times(0,1)\,,\quad R_3=(-1,0)\times(-1,0)\quad\text{and}\quad R_4=(0,1)\times(-1,0)$$

and denote

$$R=\bigcup_{i=1}^{4}R_i.$$

We extend $\left(u^0,u^1\right)$ to R in an odd fashion on R_2 and R_4 and in an even way on R_3. On the rest of the plane, we extend the data with zero value. Let us denote by $\left(\phi^0,\phi^1\right)$ these new data and consider the Cauchy problem

$$\begin{cases}\phi_{tt}-\mu\Delta\phi-(\lambda+\mu)\nabla\mathrm{div}\phi=0 & \text{in}\quad\mathbb{R}^2\times(0,T)\\ (\phi(0),\phi_t(0))=\left(\phi^0,\phi^1\right) & \text{in}\quad\mathbb{R}^2.\end{cases}\tag{P1}$$

Numerically, the solution $\phi(x,t)$ of this system may be computed in a standard way by using a FFT algorithm.

Step 2. Consider now the system

$$\begin{cases} \psi_{tt} - \mu\Delta\psi - (\lambda+\mu)\nabla\mathrm{div}\psi = 0 & \text{in} \quad \mathbb{R}^2 \times (0,T) \\ (\psi(0), \psi_t(0)) = \left(\psi^0, \psi^1\right) & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{P2}$$

where the new initial data $\left(\psi^0, \psi^1\right)$ are obtained by extending to $\mathbb{R}^2$ the restriction to Ω of the solution at time T of (P1) with a change of sign in the first component, say $(-\phi(T), \phi_t(T))|_\Omega$, in a similar fashion as in Step 1. As before, we then solve (P2).

Step 3. Next, define the initial data

$$\left(\widehat{\phi}^0, \widehat{\phi}^1\right) = \left(u^0 - \psi(T), u^1 + \psi_t(T)\right), \tag{3}$$

and with these data repeat steps 1 and 2. Let us denote by $\widehat{\phi}, \widehat{\psi}$ the corresponding solutions of (P1) and (P2), respectively. Then the function

$$\widehat{u}(x,t) = \widehat{\phi}(x,t) + \widehat{\psi}(x, T-t), \quad 0 \le t \le T, \ x \in \Omega$$

is a numerical approximation of the state $u(x,t)$ of system (1). In addition, the function

$$\widehat{v}(y,t) = \widehat{\phi}(y,t) + \widehat{\psi}(y, T-t), \qquad 0 \le t \le T, \ y \in \Gamma_1,$$

is a numerical approximation of the boundary control $v(x,t)$.

Remark 1. *As explained in detail in [6], and denoting by $X_0 \times X_1$ an appropriate function space for the initial conditions, the proof of convergence of the above described algorithm is equivalent to proving that the operator*

$$\begin{array}{rccc} L_T: & X_0 \times X_1 & \to & X_0 \times X_1 \\ & \left(\phi^0, \ \phi^1\right) & \mapsto & \left(\phi^0 + \psi(T), \ \phi^1 - \psi'(T)\right) \end{array}$$

is surjective. This amounts to showing that there exists a positive constant $C(T)$, with $C(T) < 1$, such that

$$\left\| \left(\phi|_\Omega(T), \ \phi'\big|_\Omega(T) \right) \right\|_{X_0 \times X_1} \le C(T) \left\| \left(\phi^0, \ \phi^1\right) \right\|_{X_0 \times X_1} \qquad \text{for all } \left(\phi^0, \ \phi^1\right) \in X_0 \times X_1. \tag{4}$$

For T large enough, (4) may be proved by transforming the system of elasticity into a system of wave equations and then using Poisson's formula for the wave equation (see for instance [2, 3]). Finally, we notice that the algorithm described in Steps 1-3 above is based on a first-order approximation of operator L_T^{-1}.

3. Numerical simulations

3.1. The unit square with Dirichlet controls acting on two adjacent edges

As in the preceding section we put $\Omega \equiv R_1 = (0,1)^2$ the unit square, Γ_0 the edges $x_1 = 0$ and $x_2 = 0$, and

$$\Gamma_1 = \left\{(1,s) \in \mathbb{R}^2 : \quad 0 < s \le 1, \ \right\} \cup \left\{(s,1) \in \mathbb{R}^2 : \quad 0 < s \le 1\right\}.$$

We assume that the displacement field is equal to zero at Γ_0 and that the controls act on Γ_1 as in (1). It is well-known [5, p. 474] that in this case the minimum time for exact controllability $T^* = \frac{2\sqrt{2}}{\mu}$, with μ the Poisson ratio. We take $\lambda = 0.5$, $\mu = 1$, $T = 3$ and consider the simple initial conditions

$$u^0(x_1, x_2) = (0.2\sin(\pi x_1)\sin(\pi x_2), \ 0.2\sin(\pi x_1)\sin(\pi x_2)), \quad u^1(x_1, x_2) = (0, \ 0).$$

We have used a Fast Fourier Transform (FFT) algorithm for solving the associated Cauchy problems. Precisely, following the notation of [9, Ch. 5], we have taken $N = 1024$ and $L = 32$ which provides a mesh size $h = L/N = 0.0313$ and frequency resolution $fr = 2\pi/L = 0.1963$. The grossest of aliasing errors have been removed by putting, as usual, $K = N/8$. Both the direct and inverse FFT algorithms have been tested for functions for which the Fourier transforms are explicitly known leading to errors both in the discrete L^∞ and L^2 norms of the order of $10e-14$.

Figure 1 shows the animation of the state $u(x_1, x_2, t)$ at different times and in the form of a deformed mesh.

Figure 2 displays the pictures for the controls. Precisely, denoting $v_1 = \left(v_1^1, v_1^2\right)$ the control at the edge $x_1 = 1$ and $v_2 = \left(v_2^1, v_2^2\right)$ the control at $x_2 = 1$, Figure 2 shows v_1^1 and v_2^1. Due to the symmetry of the initial data, $v_1^1 = v_1^2$ and $v_2^1 = v_2^2$. The mesh size for the time variable is equal to 0.15.

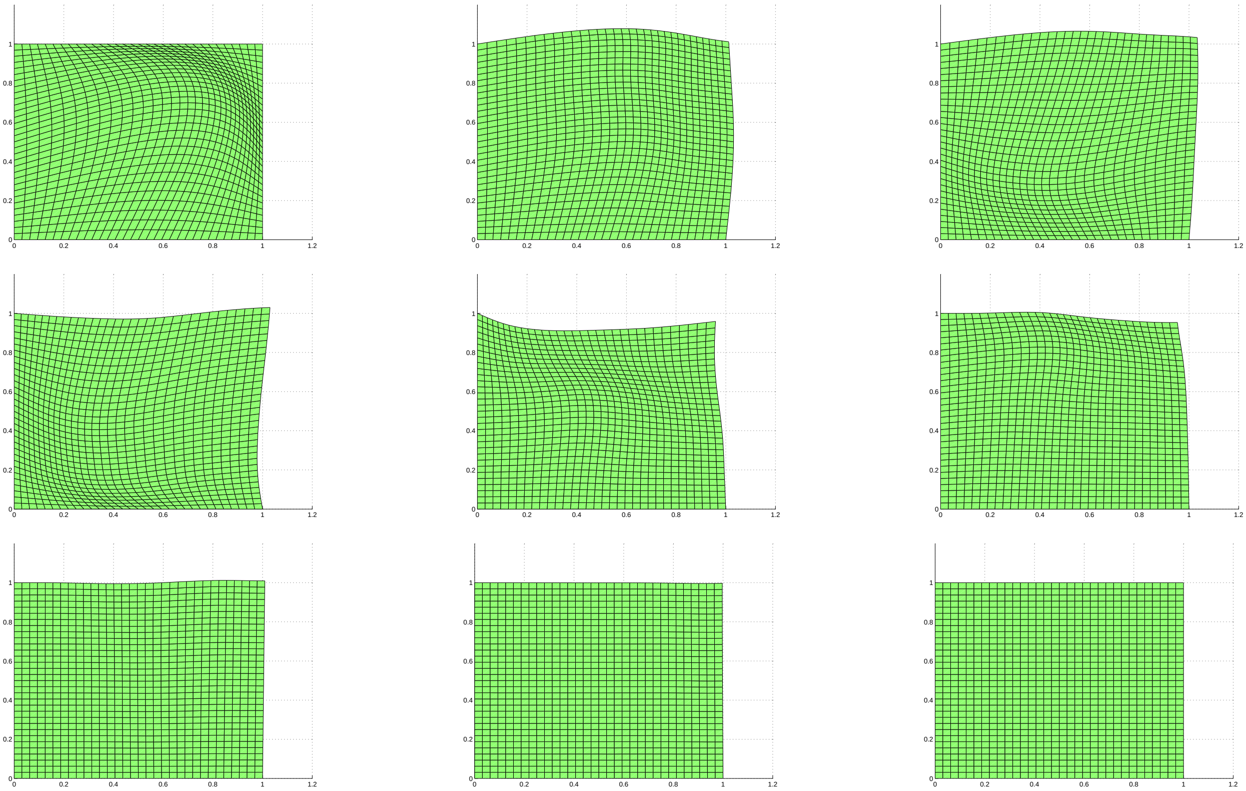


Figure 1: Animation of the state $u(x, t_k)$ from left to right and from top to bottom for $t_k = 0, 0.15, 0.30, 0.45, 0.75, 0.90, 1.65, 2.10, 3$.

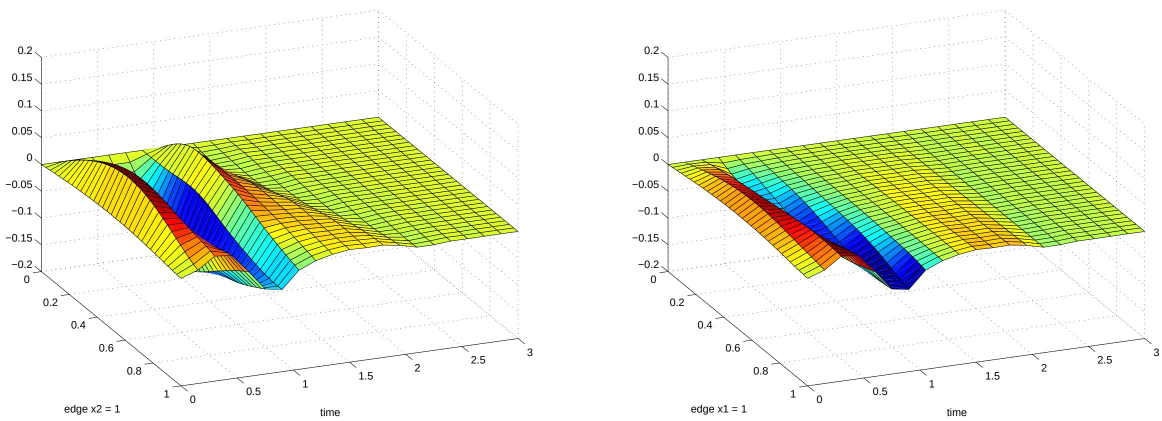


Figure 2: Picture of the controls $v_1^1(t)$ -left- and $v_2^1(t)$ -right- during the time interval $0 \leq t \leq 3$.

3.2. The unit circle with Neumann control distributed along the whole boundary

Let $\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\right\}$ be the unit circle and consider the boundary control system

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \mathrm{div} u = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ \sigma \cdot n = v & \text{on} \quad \sum = \Gamma \times (0, T) \\ (u(0), u_t(0)) = \left(u^0, u^1\right) & \text{in} \quad \Omega \end{cases} \tag{5}$$

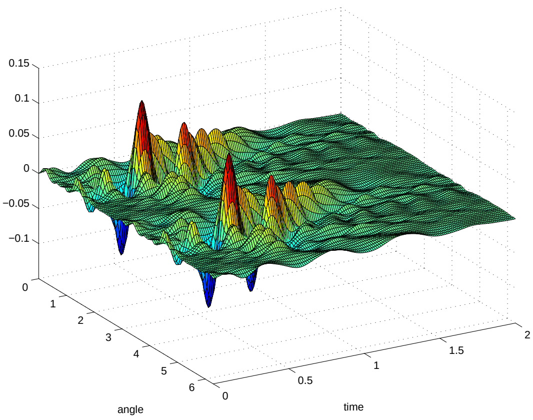
where

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \text{ with } \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \ 1 \le i, j \le 2,$$

$\lambda = 0.5$, $\mu = 1$, and $n = (n_1, n_2)$ is the outward unit normal vector to Γ. Hence, in this case the control $v = \left(v^1, \ v^2\right)$ represents a density of forces acting on the whole boundary. For the initial conditions

$$u^0(x_1, x_2) = 0.1\left(\exp\left[-64\left((x_1 - 0.2)^2 + (x_2 - 0.2)^2\right)\right], \ \exp\left[-64\left((x_1 - 0.2)^2 + (x_2 - 0.2)^2\right)\right]\right), \quad u^1(x_1, x_2) = (0, \ 0)$$

and controllability time $T = 2$, Figures 3 and 4 show the results obtained by implementing the algorithm described in the preceding section. Since, in this case the control acts on the whole boundary, the extensions of the initial conditions to $\mathbb{R}^2$ for the successive Cauchy problems have been done with zero value outside Ω. We have used the same parameters for the FFT algorithm as in the preceding case. The data for mesh sizes are $h_1 = \pi/40$ for the angle, $h_2 = 0.05$ for the radius, and $h_3 = 1/80$ for the time variable.



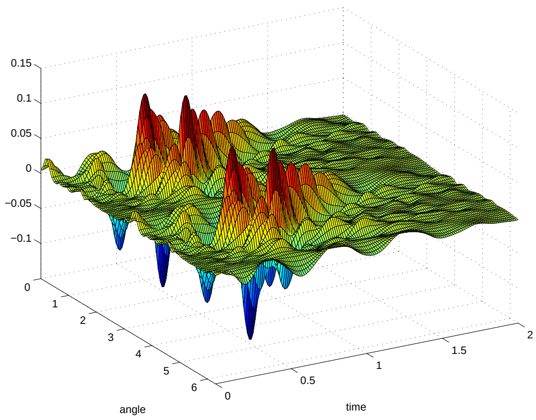


Figure 3: Pictures of the controls $v^1(t)$ -left- and $v^2(t)$ -right- for $0 \le t \le 2$.

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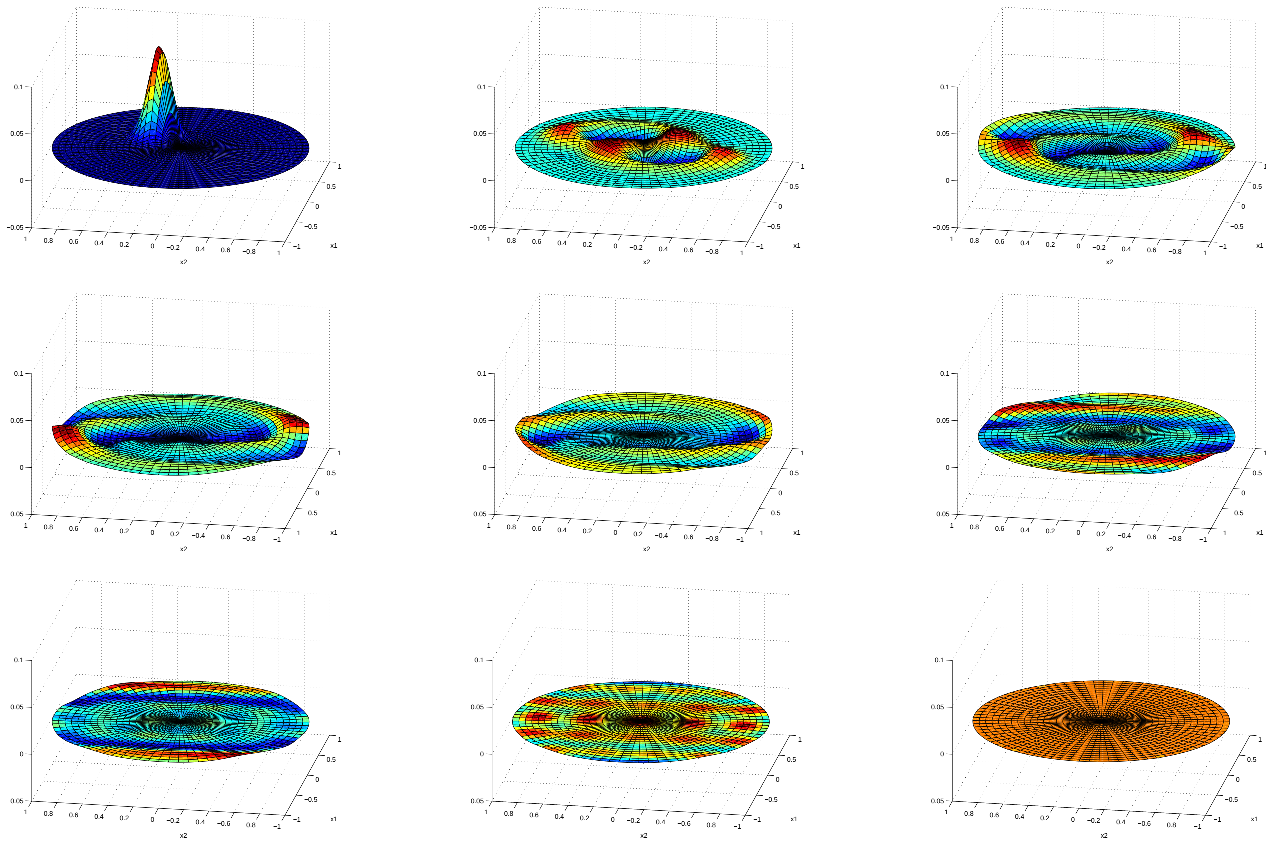

Figure 4: From left to right and from top to bottom, pictures of the first component $u_1\,(x, t_k)$ at times $t_k = 0, 0.2, 0.4, 0.5, 0.6, 0.8, 1, 1.5, 2$.