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A Numerical Method of Local Energy Decay for the Boundary Controllability of Time-Reversible Distributed Parameter Systems

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This paper deals with the numerical computation of the boundary controls of linear, time-reversible, second-order evolution systems. Based on a method introduced by Russell (*Stud. Appl. Math.* LII(3) (1973)) for the wave equation, a numerical algorithm is proposed for solving this type of problems. The convergence of the method is based on the local energy decay of the solution of a suitable Cauchy problem associated with the original control system. The method is illustrated with several numerical simulations for the Klein–Gordon and the Euler–Bernoulli equations in 1D, the wave equation on a rectangle, and the plate equation on a disk.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a regular boundary Γ . Consider the second-order evolution system

$$\begin{cases} y'' + Ay = 0, & \text{in } Q = \Omega \times (0, T) \\ y(0) = y_0, & y'(0) = y_1 & \text{in } \Omega \\ B_j y = v_j & \text{on } \Sigma = \Gamma \times (0, T) & \text{for } j = 1, \dots, m, \end{cases}$$
(1)

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where A is an elliptic operator of order 2k, k = 1, 2, ..., with constant coefficients. Typically, $A = -\Delta$, with Δ the Laplacian, $A = \Delta^2$ the biharmonic operator, or $A = -\mu\Delta - (\lambda + \mu)\nabla(\nabla \cdot)$ the operator of linear elasticity for isotropic and homogeneous materials. In this last case, the unknown y is a vector of *n*-components. As for $B_j, 1 \le j \le m$, we suppose that this is a family of linear operators acting on the spatial variable $x \in \Gamma$ for all $0 \le t \le$ T. For instance, B_j may be of Dirichlet, Neumann, or Robin type.

Given initial data (y_0, y_1) and boundary data $\{v_j\}_{1 \le j \le m}$ in appropriate spaces, let us assume that system (1) is well-posed in a suitable function space and in the sense that there exists a unique solution (in some sense) defined in the time interval [0, T]. The problem of boundary exact controllability for system (1) refers to the existence of a positive time T and a family of boundary controls $\{v_j\}_{1\le j\le m}$ such that at time T the solution of (1) satisfies the exact controllability condition

$$v(T, \cdot) = y'(T, \cdot) = 0 \quad \text{in } \Omega.$$
⁽²⁾

From a theoretical point of view, this problem has been analyzed and solved some decades ago from different perspectives. On the one hand, in the early 1970s Russell [1, 2] developed a general method which is valid not only for hyperbolic equations but also for parabolic ones. On the other hand, in the late 1980s Lions [3] introduced his famous Hilbert Uniqueness Method (HUM) in which exact controllability is deduced from a suitable observability inequality for the solutions of the associated homogeneous (i.e., uncontrolled) system.

From the numerical viewpoint, although some progress has been made in recent years, the problem of computing numerically the boundary control for general distributed parameter systems is still a challenge. The main difficulty arises in the fact that some numerical methods that are stable for solving initial-boundary value problems may be unstable in feedback stabilization and controllability because these schemes develop spurious high frequency numerical solutions that do not exist at the continuous level. It is however important to mention that since the pioneering works by Glowinski et al. [4–6] a number of sophisticated methods (most of them based on HUM) have been proposed to cure this pathology. We refer the reader to [7–11].

Returning to the method by Russell, its main idea for the case of the wave equation is to associate with the control system (1)–(2), a Cauchy problem in the whole space \mathbb{R}^n where the new initial data are defined from the original ones by extending these by zero outside Ω . Let us denote by \bar{y} the solution of this initial-value problem. Then the elements $B_j \bar{y}$, $1 \le j \le m$, acting on Γ , produce a dissipation of energy in the control region Ω where the original exact controllability problem is posed. From this, by using the superposition principle, the exact controllability condition is deduced very easily.

The aim of this work is to apply Russell's method to the numerical computation of the boundary controls of the exact controllability problem

(1)–(2). As we will see later on, the numerical scheme that we propose is very simple and easy to implement in a computer. The convergence of the algorithm is based on the fact that the operator which connects the initial data of the original control system to the restriction to Ω of the solution of its associated Cauchy problem, is contractive (see estimate (4)). This is our Theorem 1. Moreover, because the boundary controls are obtained as the sum of the restriction to Γ of the solutions of a pair of Cauchy problems, every stable numerical scheme for solving initial-value problems is also stable for computing the boundary controls. In this respect, the algorithm proposed does not generate fictitious numerical solutions.

The rest of the paper is organized as follows. In Section 2, we present a complete and detailed description of the method including its advantages and disadvantages. Section 3 is devoted to the application of the method to the linear Klein–Gordon and the Euler–Bernoulli equations in one dimension, and to the wave and plate equations in 2D. We conclude the paper with a short section on further comments and conclusions.

2. Description and analysis of the method

Let us assume that the initial data (y_0, y_1) of system (1) belong to some Banach space $X = X_0 \times X_1$.

The method we plan to describe is composed of the following three main steps:

Step 1: Extension to a Cauchy problem in the whole space. We begin by extending the initial data (y_0, y_1) of system (1) to the whole space \mathbb{R}^n . Denote by (\bar{y}_0, \bar{y}_1) this new data set. Consider now the Cauchy problem

$$\begin{cases} \bar{y}'' + A\bar{y} = 0, & \text{in } \mathbb{R}^n \times (0, T) \\ \bar{y}(0) = \bar{y}_0, & \bar{y}'(0) = \bar{y}_1 & \text{in } \mathbb{R}^n \end{cases}$$
(3)

and assume that this problem is well-posed.

Before going on with the method, a comment on the above extension of the initial data is in order. For practical reasons, it is very important to control a system acting only on a small part of the boundary. So, in general, the family of boundary controls B_jy , $1 \le j \le n$, have the form

$$B_{j}y = \begin{cases} v_{j} & \text{on} \quad \Sigma_{0} = \Gamma_{0} \times (0, T) \\ 0 & \text{on} \quad \Sigma \setminus \Sigma_{0}, \end{cases}$$

where Γ_0 is a part of Γ with positive Lebesgue measure. Thus, the extension of the initial data (y_0, y_1) should be such that the solution \bar{y} of (3) complies with the boundary conditions $B_i \bar{y} = 0$ on $\Sigma \setminus \Sigma_0$. As we will later see, in one

dimension and in simple geometries, such as rectangles in \mathbb{R}^n , this is not so difficult to obtain. If no additional conditions are imposed on the boundary controls, that is, the controls act on all of the boundary, then we propose to extend (y_0, y_1) by zero in $\mathbb{R}^n \setminus \Omega$. In both cases, the important facts are: (a) (\bar{y}_0, \bar{y}_1) have a compact support, and (b) they belong to some Banach space, say $\bar{X} = \bar{X}_0 \times \bar{X}_1$, in which system (3) is well-posed. For instance, we may think of putting $X = X_0 \times X_1$ as $H^p(\Omega) \times H^q(\Omega)$, and $\bar{X} = \bar{X}_0 \times \bar{X}_1$ as $H^p(\mathbb{R}^n) \times H^q(\mathbb{R}^n)$ for some $p, q \in \mathbb{R}$.

Notice that the restriction of \bar{y} to Ω , say $\bar{y}|_{\Omega}$, is a solution of system (1) with $v_j = B_j \bar{y}|_{\Sigma}$ for $1 \le j \le m$.

Step 2: Decay of local energy of system (3). The key point of the method is that the following local energy decay property be satisfied: There exists a positive constant C(T), with C(T) < 1, such that

$$\|(\bar{y}|_{\Omega}(T), \bar{y}'|_{\Omega}(T))\|_{X} \le C(T)\|(y_{0}, y_{1})\|_{X} \quad \text{for all } (y_{0}, y_{1}) \in X.$$
(4)

In some sense, this estimate plays a similar role as the observability inequality in the HUM. Estimate (4) has been proved for the case of the wave equation in 2D and $X = H^r \times H^{r-1}, r \ge 2$, in [1, corollary 4.2].

Step 3: Superposition's principle. In this step, we show that if (4) holds, then the exact controllability condition (2) is satisfied.

To this end, take $(\phi_0, \phi_1) \in X$ and extend these data to \mathbb{R}^n as indicated in Step 1. Let us denote by $(\bar{\phi}_0, \bar{\phi}_1) \in \bar{X}$ this new data set. Consider now the Cauchy problem

$$\begin{cases} \bar{\phi}'' + A\bar{\phi} = 0, & \text{in } \mathbb{R}^n \times (0, T) \\ \bar{\phi}(0) = \bar{\phi}_0, \quad \bar{\phi}'(0) = \bar{\phi}_1 & \text{in } \mathbb{R}^n \end{cases}.$$
(5)

Then, the function $\phi = \overline{\phi}|_{\Omega \times (0,T)}$ solves the initial-boundary value problem

$$\begin{cases} \phi'' + A\phi = 0, & \text{in } Q = \Omega \times (0, T) \\ \phi(0) = \phi_0, & \phi'(0) = \phi_1 & \text{in } \Omega \\ B_j \phi = g_j & \text{on } \Sigma = \Gamma \times (0, T) & \text{for } j = 1, \dots, m, \end{cases}$$
(6)

with $g_j = B_j \bar{\phi}$ for $j = 1, \dots, m$. Next, consider the system

$$\begin{cases} \bar{\psi}'' + A\bar{\psi} = 0, & \text{in } \mathbb{R}^n \times (0, T) \\ \bar{\psi}(0) = \bar{\psi}_0, \quad \bar{\psi}'(0) = \bar{\psi}_1 & \text{in } \mathbb{R}^n \end{cases}$$
(7)

where the initial data $(\bar{\psi}_0, \bar{\psi}_1)$ are obtained from

$$(\psi_0, \psi_1) = (-\phi(T), \phi'(T))$$
 (8)

by extending these again as in Step 1. Then, it is clear that the function $\psi = \overline{\psi}|_{\Omega \times (0,T)}$ is a solution of

$$\begin{cases} \psi'' + A\psi = 0, & \text{in } Q = \Omega \times (0, T) \\ \psi(0) = \psi_0, \quad \psi'(0) = \psi_1 & \text{in } \Omega \\ B_j \psi = h_j & \text{on } \Sigma = \Gamma \times (0, T), \quad 1 \le j \le m, \end{cases}$$
(9)

with $h_j = B_j \bar{\psi}$, $1 \le j \le m$.

Finally, set $z(x, t) = \phi(x, t) + \psi(x, T - t)$. Because the operators A and B_j are linear, z solves the exact controllability problem

$$\begin{cases} z'' + Az = 0, & \text{in } Q = \Omega \times (0, T) \\ z(0) = z_0, & z'(0) = z_1 & \text{in } \Omega \\ B_j z = f_j & \text{on } \Sigma = \Gamma \times (0, T) & \text{for} j = 1, \dots, m \\ z(T) = z'(T) = 0 & \text{in } \Omega \end{cases}$$
(10)

with

$$(z_0, z_1) = (\phi_0 + \psi(T), \phi_1 - \psi'(T))$$
(11)

and

$$f_j(x,t) = g_j(x,t) + h_j(x,T-t) \quad \text{for } (x,t) \in \Gamma \times (0,T) \quad \text{and } 1 \le j \le m.$$

This procedure gives us the desired controllability condition (2) for the initial data (11). The question now is whether every set of initial data $(y_0, y_1) \in X$ may be represented in the form (11). This is equivalent to saying that the linear operator

$$L_T: X = X_0 \times X_1 \to X = X_0 \times X_1$$
$$(\phi_0, \phi_1) \mapsto (\phi_0 + \psi(T), \ \phi_1 - \psi'(T))$$

is surjective. Decomposing $L_T = I - K_T$, where K_T is given by

$$K_T(\phi_0, \phi_1) = (-\psi(T), \ \psi'(T)),$$

it suffices to show that $||K_T|| < 1$. However, this follows directly from a two-fold application of (4). Precisely, we have

$$\|K_T(\phi_0, \phi_1)\|_X = \|(-\psi(T), \psi'(T))\|_X$$

$$\leq C(T)\|(-\phi(T), \phi'(T))\|_X$$

$$\leq (C(T))^2\|(\phi_0, \phi_1)\|_X,$$

where T is such that C(T) < 1.

As a result of the strategy we have developed in Steps 1–3, we obtain the following result.

THEOREM 1. With the same notations and assumptions as in Step 1 above, let us assume that (4) holds for some time T > 0 such that C(T) < 1. Then, for every given initial data $(y_0, y_1) \in X$ there exists a family of boundary controls $\{v_j\}_{1 \le j \le m}$ such that the solution of system (1) satisfies the exact controllability condition (2).

The method we have just described leads to the following numerical algorithm for computing the boundary controls $\{v_j\}_{1 \le j \le m}$.

2.1. Numerical algorithm

As shown in Step 3, we have obtained the exact controllability property for system (10) and for the initial data (11). So, given initial data $(y_0, y_1) \in X$, the main task is to find $(\phi_0, \phi_1) \in X$ such that $L_T(\phi_0, \phi_1) = (y_0, y_1)$. Because, $L_T = I - K_T$, with $||K_T|| < 1$, we have the following representation for the operator L_T^{-1} :

$$L_T^{-1} = I + K_T + K_T^2 + \cdots$$

Therefore, we may obtain different approximations of L_T^{-1} depending on the power of K_T that we consider. For instance, suppose that we approximate L_T^{-1} by a first-order expansion in K_T , i.e., $L_T^{-1} \approx I + K_T$, and hence

$$L_T^{-1}(y_0, y_1) \approx (y_0, y_1) + K_T(y_0, y_1).$$
 (12)

Then, we propose to follow the scheme:

- (a) Take (y₀, y₁) ∈ X, extend these data to all of ℝⁿ as in Step 1, and then solve system (5) at time t = T.
- (b) With the solution obtained in (a), consider a new pair of initial conditions as in (8), extend these data as before and then solve system (7) also at time t = T. Denote the solution of this last system by φ(x, T).
 (a) Define the new data

(c) Define the new data

$$(\phi_0, \phi_1) = (y_0 - \varphi(T), y_1 + \varphi'(T)),$$

which are first-order approximations of $L_T^{-1}(y_0, y_1)$ as in (12).

(d) With this data set, repeat steps (a) and (b), but now keep the solutions, say φ(x, t) and ψ(x, t) of these problems, and the boundary controls g_j(x, t) and h_j(x, t) as given by systems (6) and (9). Approximations of both the state y(x, t) and the family of controls {v_j}_{1≤j≤m} of system (1) are then given by

$$y^{\star}(x,t) = \phi(x,t) + \psi(x,T-t),$$
 (13)

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and

$$v_j^{\star}(x, t) = g_j(x, t) + h_j(x, T - t)$$
 for

$$(x, t) \in \Gamma \times (0, T)$$
 and $1 \le j \le m$,

respectively. Note that the approximated state (13) satisfies the initial conditions

$$\begin{cases} y^{\star}(x,0) \equiv y_0^{\star} = \phi_0 + \psi(T) \\ (y^{\star})'(x,0) \equiv y_1^{\star} = \phi_1 - \psi'(T) \end{cases}$$
(14)

which are approximations of the original ones (y_0, y_1) .

At this point, it is important to notice that if the initial data (y_0, y_1) are regular enough, then, thanks to the continuity of the trace operator, it is easy to prove a continuous dependence of the family of controls $\{v_j\}_{1 \le j \le m}$ with respect to the initial data (see [1, theorem 2.1] for the case of the wave equation). Numerically this means that the order of approximation of the controls v_j^* is the same as the one of the initial data (y_0^*, y_1^*) .

The convergence of this algorithm is a consequence of Theorem 1.

Next, we would like to make a few comments on the main advantages and disadvantages of the proposed method:

• The key point of the method is estimate (4). In the examples that we have considered, this estimate holds whenever the initial data for the extended problems have a compact support. It seems that this property is, in general, true for linear, 2*k*-order evolution systems.

Apart from this, we see:

• The main advantage of this numerical scheme is its simplicity. The controllability problem is solved with exactly the same techniques as the ones used for solving initial-value problems. In fact, as we will see in Section 3, in some simple cases explicit formulae for the solutions are available. In this respect, the convergence of the algorithm is ensured and therefore we do not have to be concerned with the presence of undesirable high-frequency, spurious, numerical solutions because they do not appear here.

• The method seems to be very general in the sense that it could be applied for general linear, time-reversible, second-order evolution systems in any dimension.

- The numerical implementation of the algorithm is easy and provides at the same time numerical approximations for both the state and the controls.
 As for *disadvantages*:
- Although in simple geometries the method can be applied to controllability 45 problems with controls acting on a small part of the boundary, for more

involved geometries the method leads to controls supported everywhere on the boundary.

• A second drawback is related to the optimal time of controllability. In the proposed method, the convergence of the algorithm depends on the time for which estimate (4) holds with a constant C(T) such that C(T) < 1. In some cases, it could be difficult to find out this optimal constant and therefore the minimal time of controllability. However, this seems not to be a big problem. As indicated in [1, p. 200], if we are able to show that for the minimal time of controllability, say T_0 , the operator K_{T_0} is compact (which in general is not so difficult to prove) and L_{T_0} is injective, then this obstacle is overcome. At least, for the examples that we will consider in Section 3, there is a numerical evidence that this is so.

3. Numerical simulations

In this section, we present several numerical experiments to illustrate the excellent performance of the scheme introduced in the preceding section.

In the four examples that follow, we consider a first-order approximation of the operator L_T^{-1} as described in (12). This reduces to a minimum the computational requirements needed to implement the algorithm and the results of the numerical tests are rather acceptable. Indeed, the Cauchy problems associated with the first three examples are solved using explicit formulae for the solutions, and thereby the computational cost basically consists in the numerical computation of a few integrals. More interesting is the plate equation example for which we do not have a solution in closed form. The corresponding Cauchy problem is then solved in a standard way with the help of the Fast Fourier Transform (FFT). Later on, we will discuss a bit more in detail the implication of this in the computational cost of the algorithm.

Finally, the computations of this section have been performed in a PC with CPU 2.66 GHz and 1.00 GB of RAM. We have used the MATLAB Toolbox and double precision.

3.1. The Klein–Gordon equation

Consider the 1D system

$$\begin{cases} y'' - y_{xx} + y = 0 & \text{in } (0, 1) \times (0, T) \\ y(x, 0) = y_0(x), \quad y'(x, 0) = y_1(x) & \text{in } (0, 1) \\ y(0, t) = 0, \quad y(1, t) = v(t) & \text{for } 0 \le t \le T, \end{cases}$$
(15)

where the control v(t) only acts at the extreme x = 1.

It is well-known ([12, p. 479]) that for regular initial data (\bar{y}_0, \bar{y}_1) , the solution of the Cauchy problem

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$$\begin{cases} \bar{y}'' - \bar{y}_{xx} + \bar{y} = 0 & \text{in } \mathbb{R} \times \mathbb{R} \\ \bar{y}(x,0) = \bar{y}_0(x), \quad \bar{y}'(x,0) = \bar{y}_1(x) & \text{in } \mathbb{R} \end{cases}$$
(16)

is given by

$$\begin{split} \bar{y}(x,t) &= \frac{\bar{y}_0(x-t) + \bar{y}_0(x+t)}{2} \\ &+ \frac{1}{2} \int_{x-t}^{x+t} \bar{y}_1(\xi) J_0(\sqrt{t^2 - (x-\xi)^2}) d\xi \\ &- \frac{t}{2} \int_{x-t}^{x+t} \bar{y}_0(\xi) \frac{J_1(\sqrt{t^2 - (x-\xi)^2})}{\sqrt{t^2 - (x-\xi)^2}} d\xi, \end{split}$$

 J_0 and J_1 being the Bessel functions of order zero and one, respectively. From this, it follows that if the initial data (\bar{y}_0, \bar{y}_1) are given by

$$\bar{y}_0(x) = \begin{cases} y_0(x), & 0 \le x \le 1\\ -y_0(-x), & -1 \le x \le 0, \\ 0, & \text{else} \end{cases} \quad \vec{y}_1(x) = \begin{cases} y_1(x), & 0 \le x \le 1\\ -y_1(-x), & -1 \le x \le 0\\ 0, & \text{else} \end{cases}$$
(17)

then the restriction of \bar{y} to [0, 1] solves (15) with $v(t) = \bar{y}(1, t)$ for t > 0.

Moreover, due to the nice asymptotic behavior of the Bessel functions [13, theorem 5.1, p. 139], it is not so hard to show that (4) holds for initial data (y_0, y_1) , for instance in $H^r \times H^{r-1}$, with $r \ge 2$, and for T large enough.

Now consider the initial data

$$y_0(x) = \begin{cases} 16x^3, & 0 \le x \le 0.5\\ 16(1-x)^3, & 0.5 \le x \le 1 \end{cases}, \quad y_1(x) = 0. \end{cases}$$

The same initial data set was considered in [7] for the case of the wave equation. Figure 1 displays the results obtained for the state (y(x, t), y'(x, t)) after the application of our algorithm for T = 2.5 and 4, respectively. The computations of the integrals appearing in the process, including the integral formulae for the Bessel functions, have been carried out with Simpson's rule. We have used a mesh size h = 0.02.

Figure 2 shows the controls for T = 2.5 and 4. As we can see, system (15) dissipates its energy mainly in the time interval [0, 2].

Finally, Table 1 presents the results for the discrete L^{∞} -norm of the difference between the initial data (y_0, y_1) and the approximated ones (y_0^*, y_1^*) s given by (14), and at different times.



Figure 1. Pictures for the states y(x, t) (left-hand panels) and y'(x, t) (right-hand panels) corresponding to system (15) for T = 2.5 (first row) and T = 4 (second row).

3.2. The Euler–Bernoulli beam equation

In this subsection, we consider the following control system for the Euler–Bernoulli beam equation:

$$\begin{cases} y'' + y_{xxxx} = 0 & \text{in } (0, 1) \times (0, T) \\ y(x, 0) = y_0(x), \quad y'(x, 0) = y_1(x) & \text{in } (0, 1) \\ y(0, t) = y_{xx}(0, t) = 0 & \text{for } 0 \le t \le T \\ y(1, t) = v_1(t), \quad y_{xx}(1, t) = v_2(t) & \text{for } 0 \le t \le T, \end{cases}$$
(18)

where now we have two controls v_1 , v_2 acting at the extreme x = 1. The goal is to choose these controls in such a way that

(y(T), y'(T)) = (0, 0) in (0, 1).

Figure 2. Pictures for the control v(t) for the Klein-Gordon system corresponding to T = 2.5 (left-hand panel) and T = 4 (right-hand panel).

Table 1

Comparison Table for the Error between the Original Initial Data Set and the Approximated One for System (15) at Different Times

Т	$\ y_0 - y_0^*\ _{\infty}$	$\ y_1 - y_1^*\ _{\infty}$
2.5	6.7×10^{-3}	2.2×10^{-3}
4	1.1×10^{-4}	2.2×10^{-3}
6	1.7×10^{-5}	9.6×10^{-5}

We proceed analogously to the Klein–Gordon system and therefore consider the initial-value problem

$$\begin{cases} \bar{y}'' + \bar{y}_{xxxx} = 0 & \text{in } \mathbb{R} \times \mathbb{R} \\ \bar{y}(x,0) = \bar{y}_0(x), \quad \bar{y}'(x,0) = \bar{y}_1(x) & \text{in } \mathbb{R}, \end{cases}$$
(19)

the initial data (\bar{y}_0, \bar{y}_1) being defined as in (17). Using the Fourier transform (see [14, pp. 23–24]), the solution of (19) can be written explicitly as

$$\begin{split} \bar{y}(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_{-1}^{1} \bar{y}_{0}(\xi) \cos\left(\frac{(x-\xi)^{2}}{4t} - \frac{\pi}{4}\right) d\xi \\ &+ \frac{1}{2\pi} \int_{-1}^{1} \bar{y}_{1}(\xi) \left\{ \pi (x-\xi) \left(S\left(\frac{x-\xi}{\sqrt{2\pi t}}\right) - C\left(\frac{x-\xi}{\sqrt{2\pi t}}\right) \right) \\ &+ \frac{1}{\sqrt{\pi t}} \sin\left(\frac{(x-\xi)^{2}}{4t} + \frac{\pi}{4}\right) \right\} d\xi, \end{split}$$

where

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x s^{-1/2} \sin s \, ds$$
 and $C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x s^{-1/2} \cos s \, ds.$

As before, the restriction of \bar{y} to the interval [0, 1] is a solution of (18) with $v_1(t) = \bar{y}(1, t)$ and $v_2(t) = \bar{y}_{xx}(1, t)$. Moreover, it is not hard to show that for p = 0, 1, 2, 3 and t > 0 there exists a constant K = K(p, t) > 0, with

$$\left|\frac{\partial^{p} y(x,t)}{\partial x^{p}}\right| \leq K(p,t) \left(\|y_{0}\|_{H^{3}(0,1)}^{2} + \|y_{1}\|_{H^{1}(0,1)}^{2}\right)$$

From this, one deduces (4) in $X = H^3(0, 1) \times H^1(0, 1)$ for T large enough. It is however well-known [15] that the exact controllability condition (2) holds for all T > 0.

Consider the simple initial data

 $K(p, t) \searrow 0$ as $t \rightarrow \infty$, such that

$$y_0(x) = \sin(\pi x), \quad y_1(x) = 0$$

Figures 3 and 4 show the pictures for the states (y(x, t), y'(x, t)) and controls $(v_1(t), v_2(t))$, respectively, for T = 2 and T = 4.

Finally, Table 2 displays the results for the error in the discrete L^{∞} -norm of the approximation of the initial data.

3.3. The wave equation

In this subsection, we focus on the wave equation in 2D. Although the 1D case has received a lot of attention in recent years, we do not treat it here because this case is very simple. In fact, explicit formulae for the controls are available (see for instance [16, p. 542]). The same holds for the 3D case thanks to Huyghens' principle [1].

Let $\Omega \equiv R_1 = (0, 1)^2$ be the unit square. We split the boundary of Ω into two parts

$$\Gamma_0 = \{(0, s) \in \mathbb{R}^2 : 0 \le s < 1, \} \cup \{(s, 0) \in \mathbb{R}^2 : 0 \le s < 1\}$$

and

$$\Gamma_1 = \{ (1,s) \in \mathbb{R}^2 : 0 < s \le 1, \} \cup \{ (s,1) \in \mathbb{R}^2 : 0 < s \le 1 \}.$$

Our main goal will be to compute the boundary control $v : \Gamma_1 \to \mathbb{R}$ such that for some fixed time $T > T_0 = 2\sqrt{2}$ (the minimal time of controllability) the solution of system



Figure 3. Pictures for the states y(x, t) (left-hand panels) and y'(x, t) (right-hand panels) corresponding to T = 2 (first row) and T = 4 (second row).

$$\begin{cases} y'' - \Delta y = 0, & \text{in } Q = \Omega \times (0, T) \\ y = 0, & \text{on } \Sigma_0 = \Gamma_0 \times [0, T] \\ y = v, & \text{on } \Sigma_1 = \Gamma_1 \times [0, T] \\ y(0) = y_0, \ y'(0) = y_1 & \text{in } \Omega \end{cases}$$
(20)

satisfies the exact controllability condition (2). At this point, we notice that in order for the exact controllability condition (2) to hold it is necessary to impose some geometrical conditions on the boundary (see for instance [17]). Our choice of the boundary control region Γ_1 fulfills such conditions.

We begin by extending the initial data (y_0, y_1) to all of \mathbb{R}^2 . To this end, consider the rectangles

$$R_2 = (-1, 0) \times (0, 1), R_3 = (-1, 0) \times (-1, 0) \text{ and } R_4 = (0, 1) \times (-1, 0)$$



Then, we introduce the functions

$$\bar{y}_0(x_1, x_2) = \begin{cases} y_0(x_1, x_2), & (x_1, x_2) \in R_1 \\ -y_0(-x_1, x_2), & (x_1, x_2) \in R_2 \\ y_0(-x_1, -x_2), & (x_1, x_2) \in R_3 \\ -y_0(x_1, -x_2), & (x_1, x_2) \in R_4 \\ 0, & \text{else} \end{cases}$$

and

$$\bar{y}_1(x_1, x_2) = \begin{cases} y_1(x_1, x_2), & (x_1, x_2) \in R_1 \\ -y_1(-x_1, x_2), & (x_1, x_2) \in R_2 \\ y_1(-x_1, -x_2), & (x_1, x_2) \in R_3 \\ -y_1(x_1, -x_2), & (x_1, x_2) \in R_4 \\ 0, & \text{else.} \end{cases}$$

With these new initial conditions consider the Cauchy problem

$$\begin{cases} \bar{y}'' - \Delta \bar{y} = 0, & \text{in } \mathbb{R}^2 \times (0, T) \\ \bar{y}(0) = \bar{y}_0, \ \bar{y}'(0) = \bar{y}_1 & \text{in } \mathbb{R}^2. \end{cases}$$

For $t > T_0$ and $x \in \Omega$, because (\bar{y}_0, \bar{y}_1) vanish outside *R*, we can use Poisson's formula to write down the solution of this system in the form

$$\bar{y}(x,t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{R} \frac{\bar{y}_{0}(\xi) d\xi}{\sqrt{t^{2} - |x - \xi|^{2}}} + \frac{1}{2\pi} \int_{R} \frac{\bar{y}_{1}(\xi) d\xi}{\sqrt{t^{2} - |x - \xi|^{2}}}.$$
 (21)

From this, it can be proved both the estimate (4) for T large enough, and the fact that the restriction of \bar{y} to Ω solves (20) for $v = \bar{y}|_{\Sigma_1}$.

Finally, consider the initial data

$$y_0(x_1, x_2) = 10 \sin(\pi x_1) \sin(\pi x_2), \quad y_1(x_1, x_2) = 0, \quad (x_1, x_2) \in \Omega.$$

Figure 5 shows the animation of the state y(x, t) for $0 \le T \le 4$. Figure 6 displays the picture of the control at the edge x = 1. We have also used Simpson's rule to compute the integrals appearing in (21). The numerical errors are similar to the two preceding cases.

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Figure 5. Animation of the state $y(x, t_k)$ from left- to right-hand panels and from top to bottom panels for $t_k = 0, 0.2, 0.6, 1.4, 2, 2.8, 3.6, 4$.



Figure 6. Picture of the control at the edge x = 1 during the time interval $0 \le T \le 4$.

3.4. The plate equation

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ be the disk of radius 1 centered at the origin, and consider the control system for the Kirchhoff plate equation

$$\begin{cases} y''(x,t) + \Delta^2 y(x,t) = 0, & t > 0, \ x \in \Omega \\ (y(x,0), y'(x,0)) = (y_0(x), y_1(x)), & x \in \Omega \\ y(x,t) = u_1(x,t), & t \ge 0, \ x \in \partial\Omega \\ \frac{\partial y}{\partial \nu}(x,t) = u_2(x,t), & t \ge 0, \ x \in \partial\Omega, \end{cases}$$
(22)

where now the controls u_1 and u_2 act on all of the boundary in the form of deflection and slope of deflection in the normal direction to the boundary.

We consider the initial conditions

$$y_0(x_1, x_2) = 0.5e^{-50((x_1 - 0.125)^2 + (x_2 - 0.125)^2)} \mathcal{X}_{\Omega}(x_1, x_2),$$

$$y_1(x_1, x_2) = 0, \quad (x_1, x_2) \in \Omega,$$

and solve numerically (22) following the method described in the preceding section. The initial conditions for the associated Cauchy problem have been obtained from (y_0, y_1) by extending these by zero outside Ω . Then, the mentioned Cauchy problem has been solved using a FFT algorithm in an standard way. Precisely, following the notation of [18, chapter 5], we have taken N = 1024 and L = 32 which provides a mesh size h = L/N = 0.0313 and frequency resolution $fr = 2\pi/L = 0.1963$. The grossest of aliasing errors have been removed by putting, as usual, K = N/8. The direct and inverse FFT algorithms have been tested for functions for which the Fourier transforms are



Figure 7. Animation of the state $y(x, t_k)$ from left- to right-hand panels and from top to bottom panels for $t_k = 0, 0.05, 0.1, 0.15, 0.4, 0.5$.

explicitly known leading to errors both in the discrete L^{∞} and L^2 norms of the order of 1.e - 14. In terms of time of computation, it requires about 7 s.

Figure 7 displays the animation of the state y(x, t) at different times and for T = 0.5. In polar coordinates, we have used a mesh size $\Delta \theta = 0.157$ for the angle, and $\Delta r = 0.1$ for the radius.

Figure 8 shows the pictures for the controls. For computing the slope of deflection control, we have used backward, first-order differences.

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- 4

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Figure 8. Pictures for the controls $u_1(t)$ (left-hand panel) and $u_2(t)$ (right-hand panel) for $0 \leq t \leq T$ and $0 \leq \theta \leq 2\pi$.

Finally, we would like to emphasize that some minor changes in the code are needed to implement other type of boundary controls, for instance a bending moment control.

4. Conclusions

It is well-known that the problem of the numerical computation of the boundary exact control for the wave equation is extremely sensitive to the numerical scheme used for this approximation. In fact, for the usual finite difference and finite element schemes convergence is known to fail.

Based on a method by D. L. Russell, we have proposed in this work a very simple algorithm for solving the problem of the numerical computation of the boundary controls for linear, second order, time-reversible distributed parameter systems. Convergence of the algorithm follows from the local energy decay of the solutions of a suitable initial-value problem associated with the original control system.

Although each of the four examples tested have been computed with a first-order approximation of the operator L_T^{-1} , an increase in the order of the approximation of this operator, for instance to second-order, only requires a recurrence in steps (a) and (b) of the numerical algorithm, which does not increase the computational cost in a significant way.

One interesting remaining point is to analyze whether the controls provide by the method presented here coincide with the ones that come from the HUM. This can be proved for the 1D wave equation, and we suspect that the same holds for more general situations. The numerical simulations presented in this work may be useful for testing numerically this conjecture.

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