4 $\bar{F}(x,\lambda,\xi) = \xi^2 + y\lambda^2$

$\frac{\partial \bar{F}}{\partial \xi} = 2\xi$; $\frac{\partial \bar{F}}{\partial \lambda} = 2y\lambda$

Ecuación de Euler-Lagrange:

$(u')' = yu$

es decir,

$$(E\text{-}L)\begin{cases} u'' - yu = 0 & \text{en }]0,\pi[\\ u(0) = u(\pi) = 0 \\ \int_0^\pi u(x)^2 dx = 1 \end{cases}$$

La solución de la EDO depende del valor de multiplicador y. Analizamos todos los posibles casos:

$\boxed{\text{Caso 1: } y = 0}$ $u''(x) = 0 \to u'(x) = C_1$

$\to u(x) = C_1 x + C_2$

$0 = u(0) = C_2$

$0 = u(\pi) = C_1\pi \to C_1 = 0$ NO.

$u(x) = 0$, pero $\int_0^\pi 0^2 dx = 0 \neq 1$

$\boxed{\text{Caso } y > 0}$

$u''(x) - y\,u(x) = 0.$

Polinomio característico:

$P(r) = r^2 - y = 0 \Rightarrow r = \pm\sqrt{y}$

solución $u(x) = c_1 e^{\sqrt{y}x} + c_2 e^{-\sqrt{y}x}$

$$\begin{cases} 0 = u(0) = c_1 + c_2 \\ 0 = u(\pi) = c_1 e^{\sqrt{y}\pi} + c_2 e^{-\sqrt{y}\pi} \end{cases}$$

$c_1 = -c_2$

$0 = c_1 \big(\underbrace{e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}}_{\neq 0} \big) \Rightarrow c_1 = 0 = c_2$

$\Downarrow$

$u(x) = 0$ (NO) pues

$\int_0^{\pi} 0^2\,dx = 0 \neq 1.$

$\boxed{\text{Caso } \lambda < 0} \longrightarrow$ Cambiar $\underline{\lambda = y}$

$P(r) = r^2 - \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda}$

solución $u(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \operatorname{sen}(\sqrt{\lambda}\,x).$

$0 = u(0) = c_1 \cancel{\cos}$

$0 = u(\pi) = c_2 \operatorname{sen}(\sqrt{\lambda}\,\pi)$ $\begin{cases} c_2 = 0 \text{ NO} \\ \operatorname{sen}(\sqrt{\lambda}\,\pi) = 0 \Leftrightarrow \end{cases}$

$\Updownarrow$

$\sqrt{\lambda}\,\pi = k\pi,\ \cancel{k\in\mathbb{N}}.k$

$\Leftrightarrow \boxed{\lambda = k^2},\ k = 1, 2, \ldots$

$$1 = \int_0^{\pi} u(x)^2 dx = \int_0^{\pi} c_1^2 \operatorname{sen}^2(kx) dx$$

$$= c_1^2 \frac{\pi}{2}$$

$$\rightarrow c_1 = \sqrt{\frac{2}{\pi}}$$

Solución:

$$u(x) = \sqrt{\frac{2}{\pi}} \operatorname{sen}(kx), \quad k = 1, 2 \ldots$$