Optimal Design of the Damping Set for the Stabilization of the Wave Equation

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Problem Formulation

We consider the nonlinear optimal design problem

$$(\mathrm{P}) \qquad \inf_{\omega \in \Omega_L} J(\mathcal{X}_\omega) = \frac{1}{2}\int_0^T \int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dx dt$$

where

- u solves the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(x)\,\mathcal{X}_\omega(x)\, u_t = 0 & \text{in} \quad (0,T)\times\Omega \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 & \text{in} \quad \Omega \end{cases}$$

  with $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, a bounded domain, $(u_0, u_1) \in H_0^1 \times L^2$, and $0 < \alpha \le a(x) < \beta < \infty$,
- $\Omega_L = \{\omega \subset \Omega : \quad |\omega| = L\,|\Omega|, \quad 0 < L < 1\}$, $|A|$ being the Lebesgue measure of A,
- $\mathcal{X}_\omega$ is the characteristic function of ω.

Theoretical Results

Relaxation

It is well-known that (P) may be ill-posed.
We then consider the relaxed problem

$$(\mathrm{RP}) \qquad \inf_{s \in S_L} J(s) = \frac{1}{2}\int_0^T \int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dx dt$$

where $S_L = \left\{ s \in L^\infty(\Omega; [0,1]) : \int_\Omega s(x)\,dx = L\,|\Omega| \right\}$ and u solves

$$\begin{cases} u_{tt} - \Delta u + a(x)\, s(x) u_t = 0, & \text{in} \quad (0,T)\times\Omega \\ u = 0, & \text{on} \quad (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 & \text{in} \quad \Omega \end{cases}$$

Main Result

Problem (RP) is a full relaxation of (P) in the sense that

(i) *there are optimal solutions for (RP),*

(ii) *the infimum of (P) equals the minimum of (RP),*

(iii) *if s is optimal for (RP) and $\mathcal{X}_{\omega_j}$ converges weakly to s, then $\mathcal{X}_{\omega_j}$ is a minimizing sequence for (P).*

Remarks

- Proof of the above result is based on a suitable reformulation of (P) as a non-convex vector variational problem and the use of gradient Young measures (see [3] and the references therein).
- Part (iii) above is equivalent to saying that the Young measure associated with an optimal sequence of damping sets is $s(x)\,\delta_1 + (1 - s(x))\,\delta_0$.
- From (iii) above it follows that the map $s \to J(s)$ is continuous for the weak* topology. This fact may also be obtained by more classical methods (see [2]), but we believe our perspective is more general in scope.

Conclusions and Perspectives

- A new approach based on the use of Young measures has been introduced for solving the nonlinear optimization problem which consists in finding the optimal damping set for the stabilization of the wave equation.
- Numerical experiments highlight the influence of the over-damping phenomena and show that for large values of the damping potencial the original problem (P) has no a minimizer. This fact justifies the relaxation's procedure.
- The methodology introduced in this work seems to be very appropriate to deal with some other more general problems like the stabilization of the system of elasticity, and dynamic optimal design problems where the designs appear in the principal part of the operator.

Numerical Results

Algorithm of minimization

Step 1: Numerical resolution of the relaxed problem (RP) by using a gradient descent method.

Step 2: Penalization's techniques to recover some elements of a minimizing sequence of (P) from the optimal relaxed density obtained in Step 1. See [1,3].

Numerical Experiments

1D case

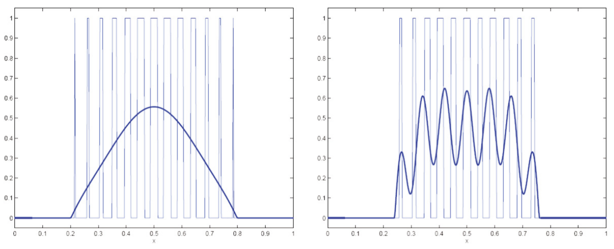

Figure 1: Optimal relaxed density and its associated bi-valued penalized density corresponding to the initial density $s_n^0(x) = L\frac{n\pi(1+\sin(n\pi x))}{n\pi + (1-\cos(n\pi))}$, for $n = 5$ (left) and $n = 25$ (right) - $L = 1/5$, $\Omega = (0,1)$, $T = 1$, $a(x) = 10\mathcal{X}_\Omega(x)$, $u_0(x) = \sin(\pi x)$, $u_1(x) = 0$.

n	5	25
$J(s_n^0)$	1.3595	1.2986
$J(s_n^r)$	1.1370	1.1354
$J(s^p)$	1.1371	1.1355

Table 1: Values of the cost function J corresponding to the densities:
(a) s_n^0 for the initialization of the algorithm,
(b) s_n^r for the optimal relaxed density, and
(c) s^p for the penalized density.

2D case

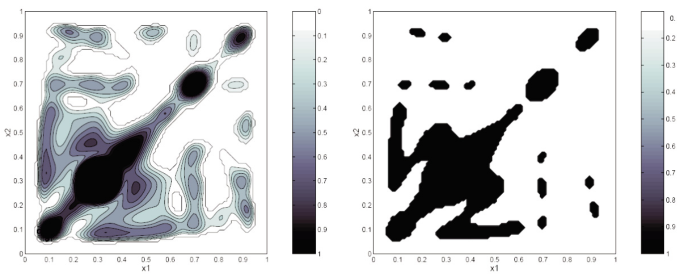

Figure 2: Optimal relaxed density s^r (left) and its associated penalized density s^p (right) corresponding to the initial density $s^0(x) = 0.2\mathcal{X}_\Omega(x)$ - $L = 0.2$ - $\Omega = (0,1)^2$ - $T = 4$ - $a(x) = 5\mathcal{X}_\Omega(x)$ - $u_0(x) = \exp^{-100(x_1-0.3)^2 - 100(x_2-0.3)^2}\mathcal{X}_\Omega, u_1(x) = 0$. The values of the cost function are $J(s^r) \approx 0.8053$ and $J(s^p) \approx 0.8257$.

References

[1] Allaire, G., *Shape optimization by the homogenization method*, Springer 2002.

[2] Fahroo, F. and Ito, K., *Variational formulation of optimal damping designs*, Contemporary Math., 209, 95-114, 1997.

[3] Münch, A., Pedregal, P. and Periago, F., *Optimal design of the damping set for the stabilization of the wave equation*. To appear in Journal of Differential Equations.

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