Favard theory for the adjoint equation and the recurrent Fredholm alternative

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The homogeneous and nonhomogeneous linear equation.

Let us consider the nonhomogeneous linear differential equation in $\mathbb{R}^n$

$$x' = A(t)\,x + f(t)\,, \qquad t \in \mathbb{R}, \tag{1}$$

where $A(t)$ and $f(t)$ are bounded and uniformly continuous.

We consider that the joint Hull $\Omega = H(A, f)$ is recurrent and we formulate (1) as a collective family

$$x' = A(\omega{\cdot}t)\,x + f(\omega{\cdot}t)\,, \qquad t \in \mathbb{R},\, \omega \in \Omega. \tag{2}$$

The solutions of (2) induce a continuous skew-product semiflow

$$\begin{aligned} \tau\colon \mathbb{R}\times\Omega\times\mathbb{R}^n &\to \Omega\times\mathbb{R}^n \\ (t,\omega,x) &\mapsto (\omega{\cdot}t, u(t,\omega,x)). \end{aligned}$$

We ask for conditions that implies the existence of bounded solutions. These conditions are given in terms of the adjoint equation

$$x' = -A^T(\omega{\cdot}t)\,x \tag{3}$$

The BP-condition.

If $x(t)$ is a solution of (1) and $v(t)$ is a bounded solution of the adjoint equation (3) then

$$
\begin{aligned}
&\int_0^t < f(s), v(s) > \, ds = \\
&< x(s), v(s) > \Big|_0^t - \int_0^t < v(s) + A^T(s)\, v(s), x(s) > \, ds = \\
&< x(t), v(t) > - < x(0), v(0) > .
\end{aligned}
$$

Then a necesary condition for the existence of a bounded solution of (1) is that $< f, v > \in BP(\mathbb{R}, \mathbb{R})$ for every bounded solution v of (3). We refer to this property as the BP-condition.

We will prove that this condition will be sufficient in some cases. We first refer to linear homogeneous equations that satisfies the Favard separation condition (F_A).

The Favard separation condition.

Given the homogeneous equation

$$x' = A(\omega{\cdot}t)\,x\,, \qquad \omega \in \Omega, \tag{4}$$

the condition (F_A) holds if for every $\omega \in \Omega$ and every bounded solution $x(t)$ of $(4)_\omega$ one has $\inf_{t\in\mathbb{R}} |x(t)| > 0$.

We denote

$$\mathcal{B} = \{(\omega, x) \in \Omega \times \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} |\phi(t,\omega)x| < \infty\} \qquad \text{and}$$

$$d(\omega) = dim\{x \in \mathbb{R}^n \mid (\omega, x) \in \mathcal{B}\}.$$

The function $d(\omega)$ i s discrete and non continuous. I t has a residual invariant subset $\Omega_F \subset \Omega$ of points with minimum value $d(\omega) = d_F(A)$ (Favard dimension) for $\omega \in \Omega_F$(minimum of d).
(F_A) holds if and only if $d(\omega) = d_F$for every $\omega \in \Omega$.
Then $\mathcal{B}$ is a continuous subbundle of $\Omega \times \mathbb{R}^n$.

The Favard Theorem of recurrence.

Theorem 1

Let us assume that the nonhomogeneous equation (2) admits a bounded solution, then

(i) the equation (2) has a recurrent solution $x(t) = u(t, \omega_0, x_0)$ such that $H(x)$ is a minimal almost automorphic extension of (Ω, σ).

(ii) If (F_A) holds, the equation (2) has a recurrent solution $x(t) = u(t, \omega_0, x_0)$ such that $H(x)$ is a copy of the base (Ω, σ).

In this context, several possible assertions can be analized:

(1) If (F_A) holds then (F_A^*) does.

(2) If (F_A) and (F_A^*) hold then $d_F(A) = d_F^*(A)$.

(3) If (F_A) and (F_A^*) hold and $d_F(A) = d_F^*(A)$ then the BP-condition is sufficient to obtain bounded solutions.

The Sacker-Sell spectral theory.

A second ingredient to give a positive answer to (3) is given in terms of the Sacker-Sell spectrum of A.
The spectrum $\sigma(A)$ is the set of the real λ's for which the homogeneous equation

$$x' = [A(\omega{\cdot}t) - \lambda I]\, x$$

does not admit exponential dichotomy on $\mathbb{R}$. This is always a nonempty compact set defined at most n adjoint closed intervals

$$\sigma(A) = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_m, b_m]\,, \qquad m \leq n.$$

We denote by $d_S(A)$ (Sacker-Sell dimension) the vectorial dimension of the subbundle associated to the spectral interval that contains 0.
The bounded solutions are included in this subbundle, so we always have

$$d_F(A) \leq d_S(A).$$

The main theorems.

Theorem 2

Assume that $H(A, f)$ is minimal and $d_F(A) = d_S(A)$. Then (F_A) and (F_A^*) hold with $d_F(A) = d_F^*(A)$ and condition BP is sufficient to obtain bounded solutions of the nonhomogeneous equation (2).

A comment is done about condition BP: when A and f are T-periodic the conclusions of the theorem are true even if $d_F(A) = d_S(A)$ is not satisfied. However our problem here is different: we try to obtain the best conclusions of every nonhomogeneous term. Next result shows that this difference is crucial.

Theorem 3

Assume $d_S(A) \leq 2$ and (F_A) and (F_A^*) hold. If condition BP is sufficient to obtain bounded solutions of the nonhomogeneous equation (2) for every f such that $H(A, f)$ is minimal then $d_F(A) = d_S(A)$.

Other versions of the problem in the literature.

J. K. Hale, *Ordinary differential equations*, Wiley, (1969) posed this problem when $A(t)$ is purely periodic and $f(t)$ is almost-periodic (A-P for short).

Proposition 4

Let $A(t)$ be a continuous and periodic function. Assume that for every recurrent f, the nonhomogeneous equation (2) has a bounded solution if and only if the BP-condition holds then $d_F(A) = d_S(A)$.

K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Diff. Eq., (1984).
He studies a close problem when $A(t)$ and $f(t)$ are just bounded and continuous assuming exponential dichotomy in $\mathbb{R}^+$ and $\mathbb{R}^-$ of the homogeneous linear equation.

P. Cieutat and A. Haraux, *Exponential decay and existence of almost-periodic solution for some linear forced differential equations*, Port. Math., (2002).

They consider $A(t)$, $f(t)$ A-P functions, $A(t)$ with sign, for instance $A(t) \geq 0$. Here $A(t)$ is positive asymmetric and the sign is that of its symmetric part $S_A(t) = \frac{A(t)+A^T(t)}{2}$. They also assume that the antisymmetric part $K_A(t) = \frac{A(t)-A^T(t)}{2}$ is purely periodic. They prove that the nonhomogeneous equation has an A-P solution if and only if $\int_0^t < f(s), v(s) > ds$ is A-P for every A-P solution of the pair of conditions

$$v' = K_A(t)\, v\,, \qquad S_A(t)\, v(t) = 0.$$

Theorem 5

Assume that $A(t)$ and $f(t)$ are jointly recurrent and the matrix $A(t)$ has sign. Then $d_F(A) = d_S(A)$ and the nonhomogeneous equation admits bounded solution if and only if BP holds.

Robustness of the BP-condition.

R. Ortega, M. Tarallo, *Almost-periodic equations and conditions of Ambrosetti-Prodi type*, Math. Proc. Camb. Phil. Soc., (2003).
They consider the recurrent damped Hill equation

$$x'' + c\,x' + a(t)\,x = g(t)\,,$$

with $c \neq 0$, $a(t)$ and $g(t)$ are A-P, whose homogeneous part is disconjugated in a strong sense. They show that BP-condition implies the existence of bounded A-P solutions.
But here is possible to prove that $d_F(A) = d_S(A) = 1$.
We need a preliminary result that provides the correct formulation of the BP-condition, assuming (F_A^*).

Lemma 6

Assume that Ω is minimal and (F_A^*) holds. If

$$< f_\omega, \phi_A^*(\cdot,\omega)\,\xi > \in BP \qquad \forall \xi \in \mathcal{B}_\omega^*(A)$$

holds for some $\omega_0 \in \Omega$. Then it holds for every $\omega \in \Omega$.

The proof of the statement

The Cauchy operator of the adjoint equation is

$$\Phi^*_A(t,\omega) = \phi_{-A^T}(t,\omega) = \{\phi_A(t,\omega)^T\}^{-1}.$$

Using the minimality we find $\omega_0 \cdot t_n = \omega$. The map

$$L\colon \mathcal{B}^*_{\omega_0}(A) \to \mathcal{B}^*_{\omega}(A)\,, \qquad \xi_0 \mapsto \lim_{n\to\infty} \phi^*_A(t_n,\omega_0)\,\xi_0$$

is an isomorphism. Take $\xi_0 = L^{-1}\xi$, by hipothesis

$$\Big|\int_0^t < f(\omega_0\cdot s), \phi^*_A(s,\omega_0)\,\xi_0 > \, ds\Big| \le M \qquad \text{for every } t\in\mathbb{R}.$$

The proof of the statement

The cocycle identity provides

$$\begin{aligned}
&\Big| \int_0^t < f((\omega_0 \cdot t_n) \cdot s), \phi_A^*(s, \omega_0 \cdot t_n)\, \phi_A^*(t_n, \omega_0)\, \xi_0 > \, ds \Big| \\
&= \Big| \int_0^t < f(\omega_0 \cdot (s + t_n)), \phi_A^*(s + t_n, \omega_0)\, \xi_0 > \, ds \Big| \\
&= \Big| \int_0^{t+t_n} < f(\omega_0 \cdot s), \phi_A^*(s, \omega_0)\, \xi_0 > ds \\
&\quad - \int_0^t < f(\omega_0 \cdot s), \phi_A^*(s, \omega_0)\, \xi_0 > ds \Big| \leq 2M
\end{aligned}$$

and hence

$$\Big| \int_0^t < f(\omega \cdot s), \phi_A^*(s, \omega)\, \xi > \, ds \Big| \leq 2M$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}$.

The conditions $(F_A) + [d_S = d_F]$

R. J. Sacker, G. R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems III*, J. Diff. Eq., (1976).
They assume that (F_A) holds. Thus $\mathcal{B}(A)$ is an invariant subbundle.
A flow can be defined in $\mathcal{B}(A)^{\perp}$ by projecting the operator ϕ_A.
The second assumption is that this induced flow has not bounded solutions but the trivial one. Sacker and Sell proves that these assumptions are equivalent to the existence of a trichotomy. That is, the stable and unstable fibers spaces $\mathcal{U}(A)$ and $\mathcal{S}(A)$ are also subbundles and moreover

$$\Omega \times \mathbb{R}^n = \mathcal{U}(A) \oplus \mathcal{B}(A) \oplus \mathcal{S}(A)$$

This implies $d_F = d_S$. The converse is consequence of the Spectral Theorem proved in
R. J. Sacker, G. R. Sell, *A spectral theory for linear differential systems*, J. Diff. Eq., (1978).

The maximal and minimal dimensions.

Proposition 7

Let us assume $d_F(A) = n$. The $\sigma(A) = \{0\}$ and (F_A^*) holds with $d_F^*(A) = n$. If the BP-condition is satisfied for a jointly recurrent term f, the nonhomogeneous equation admit bounded solutions.

Let us fix $\omega \in \Omega$. There exist constants $0 < m \leq M < +\infty$ such that

$$m\,|\xi| \leq |\phi_A(t,\omega)\,\xi| \leq M\,|\xi| \qquad \text{and}$$
$$\frac{1}{M}|\xi| \leq |\phi_A^*(t,\omega)\,\xi| \leq \frac{1}{m}\,|\xi|$$

for every $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In particular (F_A^*) holds with $d_F^*(A) = n$.

Assume now that BP is satisfied and observe that

$$< \int_0^t \phi_A^{-1}(s,\omega)\,f(\omega\cdot s)\,ds, \xi >= \int_0^t < f(\omega\cdot s), \phi_A^*(s,\omega)\xi > \,ds$$

for every $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The left integral is bounded in $t \in \mathbb{R}$

Change of variables and Fredholm alternative.

It requires some consecutive steps. The first step is consider an epimorphism of minimal flows $\varphi\colon \Theta \to \Omega$ and the new equation

$$z' = \mathrm{A}\circ\varphi(\theta\cdot t)\, z\,, \qquad \theta\in\Theta \tag{5}$$

We say that $\mathcal{A} = A\circ\varphi$ extends A and write $\mathcal{A} \succ A$ and $\Theta \succ \Omega$.
A Lyapunov-Perron transformation on Θ is a map $Q \in C(\Theta, GL(n))$ such that DQ exists and is also continuous.
The change of variables $z = Q(\theta\cdot t)\, u$ transform (5) into

$$u' = E(\theta\cdot t)\, u$$

where $E(\theta) = Q(\theta)^{-1}\{A(\varphi(\theta))\, Q(\theta) - DQ(\theta)\}$. E is called a minimal kinematic extension of A and write $E > A$. This is the second step of the process.
When φ is an isomorphism we talk about kinematic similarity writing $E \sim A$.

Definition 8

We say that $A \in C(\Omega, \mathcal{L}(n))$ has the property (C_A) when whatever $f \in C(\Omega, \mathbb{R}^n)$ we take that if the condition

$$< f_\omega, \phi_A^*(\cdot,\omega)\,\xi > \in BP(\mathbb{R},\mathbb{R})\,, \qquad \forall \xi \in \mathcal{B}_\omega^*(A) \tag{6}$$

is satisfied for every $\omega \in \Omega$, then the equation

$$x' = A(\omega{\cdot}t)\,x + f(\omega{\cdot}t) \tag{7}$$

admits bounded solutions for every $\omega \in \Omega$.

Definition 9

Let (Ω, σ) be a minimal flow and $A \in C(\Omega, \mathcal{L}(n))$. We say that A has the recurrent Fredholm Alternative property when

(a) Conditions (F_A) and (F_A^*) are satisfied.

(b) Every minimal extension $\mathcal{A} \succ A$ satisfies $(C_{\mathcal{A}})$.

Lemma 10

Assume $\mathcal{A} \succ A$. If $(C_{\mathcal{A}})$ holds then (C_A) holds too.

Write $\mathcal{A} = A \circ \varphi$ where $\varphi\colon \Theta \to \Omega$ is an epimorphism. Take $f \in C(\Omega, \mathbb{R}^n)$ and suppose that BP-condition (6) is satisfied. Since $f \circ \varphi \in C(\Theta, \mathbb{R}^n)$ we are in the scope of condition $(C_{\mathcal{A}})$ to conclude set $\varphi(\theta) = \omega$, then

$$(f \circ \varphi)_\theta = f_\omega\,, \qquad \phi^*_{A\circ\varphi}(t,\theta) = \phi^*_A(t,\omega)\,, \qquad \text{and}$$
$$z' = (\mathrm{A} \circ \varphi)(\theta \cdot t)\, z + (f \circ \varphi)(\theta \cdot t)$$

is just our nonhomogeneous equation.

Lemma 11

Assume that $E \sim A$. Then (C_E) is equivalent to (C_A).

Let $\varphi\colon \Theta \to \Omega$ be an isomorphism and $Q\colon \mathbb{R} \to GL(n)$ the Lyapunov-Perron transformation. Condition (C_E) refers to the existence of bounded solutions of

$$u' = E(\theta\cdot t)\, u + g(\theta\cdot t) \tag{8}$$

Take f with $g(\theta) = Q(\theta)^{-1}\, f(\varphi(\theta))$ and consider ω with $\varphi(\theta) = \omega$. The change of variables $x = Q(\theta\cdot t)\, u$ takes (7) into (8) and defines a bijection between their bounded solutions. The change $y = Q^*(\theta\cdot t)\, v$ with $Q^*(\omega) = (Q(\omega)^{-1})^T$. Moreover

$$\begin{aligned} < g(\theta\cdot t), v(t) > &= < Q(\theta\cdot t)^{-1}\, f(\varphi(\theta\cdot t)), v(t) > \\ &= < f(\omega{\cdot}t), Q^*(\theta\cdot t)\, v(t) > \\ &= < f(\omega{\cdot}t), y(t) > . \end{aligned}$$

Proposition 11

Let $E > A$ be a given minimal kinematic extension. The two following facts are equivalent

(1) every minimal extension $\mathcal{A} \succ A$ satisfies $(C_{\mathcal{A}})$

(2) every minimal extension $\mathcal{E} \succ E$ satisfies $(C_{\mathcal{E}})$

Let $\varphi\colon \Theta \to \Omega$ be the epimorphism underlying $B > A$ and $Q\colon \Omega \to GL(n)$ the Lyapunov-Perron transformation allowing to write E from A.

$(1) \Rightarrow (2)$ Write $\mathcal{E} = E \circ \psi$ with $\psi : \Sigma \to \Theta$ an epimorphism. Then

$$\mathcal{E} = E \circ \psi \sim (A \circ \varphi) \circ \psi = A \circ (\varphi \circ \psi) = \mathcal{A} \succ A.$$

$(2) \Rightarrow (1)$ Write $\mathcal{A} = A \circ \psi$ where $\psi\colon \Sigma \to \Omega$ is an epimorphism. Consider $\Theta \times \Sigma$ and denote by p and q the projections on the factors. The subset $\{(\theta, \sigma) \in \Theta \times \Sigma \mid \varphi(\theta) = \psi(\sigma)\}$ contains a minimal set M. Moreover $\varphi \circ p = \psi \circ q$ holds in M. Then

$$\mathcal{E} = E \circ p \sim (A \circ \varphi) \circ p = A \circ (\varphi \circ p) = (A \circ \psi) \circ q = \mathcal{A} \circ q \succ \mathcal{A}$$

and the conclusion follows from the previous Lemma.

The direct Theorem.

Theorem 12

Assume that Ω is minimal and $A \in C(\Omega, \mathcal{L}(n))$. If

$$d_F(A) = d_S(A)$$

then A has the recurrent Fredholm Alternative property and $d_F(A^*) = d_S(A^*)$

We can consider $0 < d_S(A) = m < n$ and $k = n - m$. Let $\mathcal{V}(A)$ be the spectral subbundle corresponding to the spectral containing 0. That is $\mathcal{B}(A) = \mathcal{V}(A)$. Consider now the SS-spectral decomposition

$$\Omega \times \mathbb{R}^n = \mathcal{V}(A) \oplus \mathcal{W}(A)$$

where $\mathcal{W}(A)$ is the direct sum of the direct subbundles corresponding to the spectral intervals in $\sigma(A) - \{0\}$. The papers

K. J. Palmer, On the reducibility of almost-periodic systems of linear systems, *J. Diff. Eq.*, (1980)

R. Ellis, R. A. Johnson, Topological dynamics and linear differential systems, *J. Diff. Eq.*, (1982) Show that $\mathcal{V}(A)$ and $\mathcal{W}(A)$ can be untwisted on a minimal flow by a kinetic extension $E > A$. That is E is block-diagonal

$$E = \begin{bmatrix} E_{\mathcal{V}} & 0 \\ 0 & E_{\mathcal{W}} \end{bmatrix},$$

with blocks $E_{\mathcal{V}}$ and $E_{\mathcal{W}}$ having dimensions m and k respectively. The solutions of the two uncoupled system

$$\begin{cases} v' = E_{\mathcal{V}}(\theta \cdot t)v \\ w' = E_{\mathcal{W}}(\theta \cdot t)w, \end{cases} \tag{9}$$

are (modulo the change of variables) the solutions of the original systems that lie in $\mathcal{V}(A)$ and $\mathcal{W}(A)$ respectively. We have

$$d_F(E_{\mathcal{V}}) = m\,, \qquad 0 \text{ is not in } \sigma(E_{\mathcal{W}}).$$

We consider a flow epimorphism $\Psi : \Sigma \to \Theta$ and $\mathcal{E} = E \circ \Psi > E$ defines a linear system similar to (9). Thus is sufficient to prove the (C_E) property for (9) with adjoint equation

$$\begin{cases} v' = -E_{\mathcal{V}}(\theta t)v \\ w' = -E_{\mathcal{W}}(\theta t)w, \end{cases} \tag{10}$$

A solution of (10) is bounded if and only if v is bounded and $w = 0$. we have $d_F(E_{\mathcal{V}}) = m = d^*_F(E_{\mathcal{V}})$, thus (F^*_E) holds with dimension m. Since these conditions are invariant by kinematic extension also (F^*_A) holds with $d^*_F(A) = m$.
Consider $f \in C(\Theta, R^n)$ and decompose it as $f = (g, h)$. The BP-condition

$$< g_\theta, \phi^*_{E_{\mathcal{V}}}(\cdot, \theta)\, \xi >\in BP(\mathbb{R}, \mathbb{R})\,, \qquad \forall \xi \in \mathcal{B}^*_\theta(A)$$

is now equivalent to the existence of bounded solutions of $v' = E_{\mathcal{W}}(\theta t)v + g(\theta t)$. We denote by v one of these solutions and take W, the unique bounded solution of $w' = E_{\mathcal{W}}(\theta t) + h(\theta t)$. Then (v, w) is bounded solution of the nonhomogeneous equation, concluding the proof of (C_E). A notion of A-P Fredholm Alternative can be introduced when (Ω, σ) is an A-P flow and f is A-P. these theorems assure the existence of A-P solutions, however their proof is outsside of the A-P framework. The reason is that the extension $\Theta > \Omega$, where A diagonalizes by blocks, may fail to be A-P even if (Ω, σ) is.

R. A. Johnson, On a Floquet theory for almost-periodic two-dimension linear systems, *J. Diff. Eq.*, (1980),
R. A. Johnson, K. Palmer, G. R. Sell, Ergodic properties of linear dynamical systems, *SIAM J. Math. Anal.*, (1981)
proves the existence of a kinematic extension of A, $E > A$ such that

$$x' = E(\omega{\cdot}t)\, x$$

has a triangular form. For $n = 2$ we consider

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a(\omega{\cdot}t) & b(\omega{\cdot}t) \\ 0 & c(\omega{\cdot}t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

with adjoint equation

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -a(\omega{\cdot}t) & 0 \\ -b(\omega{\cdot}t) & -c(\omega{\cdot}t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{12}$$

Lemma 13

Conditions (F_E) and $({F_E}^*)$ hold simultaneously if and only if (F_a) and (F_c) do the same. In this case $d_F^*(E) = d_F(E)$.

(F_a) and (F_c) refer to the one-dimensional systems $x' = a(\omega{\cdot}t)\, x$ and $x' = c(\omega{\cdot}t)\, x$.

The converse Theorem

Theorem 14

Assume that $0 \in \sigma(E)$. The conditions (F_E) and (F_E^*) are jointly satisfied if and only if E is kineticaly similar on Ω to either

$$A_* = \begin{bmatrix} a_* & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{with } 0 \notin \sigma a_* \tag{13}$$

or to

$$E_* = \begin{bmatrix} 0 & b_* \\ 0 & 0 \end{bmatrix} \tag{14}$$

where a_* and $b_* \in C(\Omega, \mathbb{R})$.

It is clear that A_* and E_* satisfy the direct and adjoint Favard conditions. Assume now that (F_E) and (F_E^*) are satisfied and hence also (F_a) and (F_c).

Since $0 \in \sigma(E) = \sigma(a) \cup \sigma(c)$ we can distinguish three cases:

(1) If $0 \notin \sigma(a)$, $c \in BP(\Omega)$ it is possible to construct $\hat{c}$, $p \in C(\Omega)$ with

$$D\hat{c} = c\,, \qquad Dp = a\,p + b\,e^{\hat{c}}$$

A direct computation shows that the change of variables $x_1 = u_1 + p(\omega \cdot t)\,u_2$, $x_2 = e^{\hat{c}(\omega \cdot t)}\,u_2$ takes (11) into (13) with $a_* = a$.

(2) If $a \in BP(\Omega)$, $0 \notin \sigma(c)$ the previous arguments can be applied to the adjoint system. After swapping the two components and taking the adjoint we deduce that E is kinetically similar to A_* with $a_* = c$.

(3) If $a \in BP(\Omega)$, $c \in BP(\Omega)$ the diagonal change of variables $x_1 = e^{\hat{a}(\omega \cdot t)}\,u_1$ and $x_2 = e^{\hat{c}(\omega \cdot t)}\,u_2$ takes (11) into (14) with $b_* = b\,e^{\hat{c}-\hat{a}}$.

Proposition 15

Let Ω minimal and A_* given by (13), then

$$\sigma(A_*) = \{0\} \cup \sigma(a_*)\,, \qquad d_F(A_*) = 1 = d_S(A_*)$$

and A_* has the recurrent Fredholm Alternative property.

Proposition 16

Let Ω be minimal and E_* given by (14), then

$$\sigma(E_*) = \{0\}\,,\; d_S(E_*) = 2\,,\; d_F(E_*) = \begin{cases} 1 & \text{if } b_* \notin BP(\Omega) \\ 2 & \text{if } b_* \in BP(\Omega) \end{cases}$$

and E_* has the recurrent Fredholm Alternative property if and only if $b_* \in BP(\Omega)$.

Note that $\sigma(E_*) = 0$ and $d_S(E_*) = 2$. The general solution of (14) $x_1 = x_{10} + x_{20} \int_0^t b_*(\omega \cdot s)\, ds\,,\; x_2 = x_{20}$ and $y_1 = y_{10}\,,\; y_2 = y_{20} - y_{10} \int_0^t b_*(\omega \cdot s)\, ds$ for the adjoint equation. The theorem applies when $b_* \in BP(\Omega)$. We next consider the case $b_* \notin BP(\Omega)$. Given $f,\, g \in C(\Omega)$ and the equations

$$x_1' = b_*(\omega{\cdot}t)\, x_2 + f(\omega{\cdot}t)\,, \qquad x_2' = g(\omega{\cdot}t)$$

Here $g \in BP(\Omega)$ is the BP-condition. Suppose it and take $\hat{g} \in C(\Omega)$ with $D\hat{g} = g$. Then $x_2 = x_{20} + \hat{g}(\omega \cdot t)$ and hence

$$x_1' = b(\omega_*)\{x_2 + \hat{g}(\omega \cdot t)\} + f(\omega \cdot t)$$

and the existence of bounded solutions writes as

$$b_*(x_{20} + g\hat{\ }) + f \in BP(\Omega)$$

for suitable x_{20}. This can be satisfied when Ω is periodic. When Ω is aperiodic and $g = \hat{g} = 0$ it is possible to choose f with $\lambda b_* + f \notin BP$ for every λ. This implies no bounded solutions.

Theorem 17

Assume that Ω is minimal and $d_S(A) \leq 2$. If A has the recurrent Fredholm Alternative then $d_F(A) = d_S(A)$. Assume $0 \in \sigma(A)$ and $0 < m = d_S(A) < n$. Let E be the kinematic extension with

$$E = \begin{bmatrix} E_{\mathcal{V}} & 0 \\ 0 & E_{\mathcal{W}} \end{bmatrix},$$

then $0 \in \sigma(E_{\mathcal{V}})$, $0 \notin \sigma(E_{\mathcal{W}})$, and hence $d_S(A) = d_S(E_{\mathcal{V}})$, $d_F(A) = d_F(E_{\mathcal{V}})$.

The proof of the converse condition.

The Fredholm Alternative is valid for $E_{\mathcal{V}}$ and the conditions $(F_{E_{\mathcal{V}}})$ and $(F^*_{E_{\mathcal{V}}})$ hold.
Let us finally use the assumption $1 \leq m \leq 2$ in connection with the fact that $E_{\mathcal{V}}$ has the Fredholm alternative. Since $0 \in \sigma(E_{\mathcal{V}})$ and $(F_{E_{\mathcal{V}}})$ is satisfied when $m = 1$ one has $1 = d_F(E_{\mathcal{V}}) = d_F(A)$.
Assume now $m = 2$. since $(F^*_{E_{\mathcal{V}}})$ is satisfied then $(E_{\mathcal{V}})$ is kinematically similar to (13) or (14).
But A_* has to be excluded since $d_S(A_*) = 1$ and we are in the case $d_S(A) = d_S(E_{\mathcal{V}}) = 2$. Thus $E_{\mathcal{V}}$ must be kinematically similar to E_*. Since E_* inherits the recurrent Fredholm Alternative from E, Proposition 16 guarantees that we are in the case $d_F(E_*) = 2$.
Hence the desired onequality $2 = d_F(E_*) = d_F(A) = d_S(A)$ is satisfied. Thus the proof of the theorem is complete.