Atractores en EDPs parabólicas, escalares y casi-periódicas con exponente 0

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Starting point

CARDOSO, LANGA, OBAYA, Characterization of cocycle attractors for nonautonomous reaction-diffusion equations, *Internat. J. Bifur. Chaos*, **26** (8) (2016).

CARABALLO, LANGA, OBAYA, Pullback, forward and chaotic dynamics in 1-D non-autonomous linear-dissipative equations, *Nonlinearity* **30** (1) (2017).

Scalar ODEs $\quad y' = h(p{\cdot}t)\, y + g(y) \quad$ with null exponent.

Our theory:

CARABALLO, LANGA, OBAYA, S., Global and cocycle attractors for non-autonomous reaction-diffusion equations. The case of null upper Lyapunov exponent, *J. of Differential Equations* **265** (2018).

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Family of scalar linear-dissipative parabolic PDEs over P

Given a minimal, uniquely ergodic and aperiodic flow $(P, \cdot, \mathbb{R})$ over a compact metric space P, for each $p \in P$ we consider the problem for $y(t,x)$ with Neumann or Robin boundary conditions:

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta\, y + h(p{\cdot}t, x)\, y + g(p{\cdot}t, x, y)\,, \quad t > 0\,,\; x \in U, \\ \alpha(x)\, y + \dfrac{\partial y}{\partial n} = 0\,, \quad t > 0\,,\; x \in \partial U, \end{cases}$$

where:

$x \in \bar{U} \subset \mathbb{R}^m$ $(m \geq 1)$ the spatial domain;

$h : P \times \bar{U} \to \mathbb{R}$ determines the linear term;

$g : P \times \bar{U} \times \mathbb{R} \to \mathbb{R}$ is the dissipative term;

$\alpha : \partial U \to \mathbb{R}$ is sufficiently regular and nonnegative;

$\partial/\partial n$ is the outward normal derivative at the boundary.

We start with the dynamics in the linear case

Associated linear problems, for each $p \in P$,

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta\, y + h(p{\cdot}t, x)\, y\,, & t > 0\,,\ \ x \in U, \\ \alpha(x)\, y + \dfrac{\partial y}{\partial n} = 0\,, & t > 0\,,\ \ x \in \partial U, \end{cases}$$

where $h \in C(P \times \bar{U})$.

To the IBV problem for each $p \in P$ and $z \in C(\bar{U})$,

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta\, y + h(p{\cdot}t, x)\, y\,, \quad t > 0\,,\ x \in U, \\ \alpha(x)\, y + \dfrac{\partial y}{\partial n} = 0\,, \quad t > 0\,,\ x \in \partial U, \\ y(0, x) = z(x)\,, \quad x \in \bar{U}, \end{cases}$$

we associate an abstract linear Cauchy problem in $C(\bar{U})$,

$$\begin{cases} v'(t) = A\, v(t) + \tilde{h}(p{\cdot}t)\, v(t)\,, \quad t > 0\,, \\ v(0) = z\,, \end{cases} \tag{1}$$

with $\tilde{h} : P \to C(\bar{U})$, $\tilde{h}(p)(x) = h(p, x)$ for $x \in \bar{U}$.

This problem has a unique *mild solution*: a continuous map $v(t) = v(t, p, z) : [0, \infty) \to C(\bar{U})$ which satisfies the integral equation

$$v(t) = e^{tA} z + \int_0^t e^{(t-s)A}\, \tilde{h}(p{\cdot}s)\, v(s)\, ds\,, \quad t \geq 0\,.$$

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The linear skew-product semiflow induced by mild solutions

Mild solutions induce a globally-defined continuous linear skew-product semiflow

$$\begin{array}{rccc} \tau_L: & \mathbb{R}_+ \times P \times C(\bar{U}) & \longrightarrow & P \times C(\bar{U}) \\ & (t,p,z) & \mapsto & (p{\cdot}t, \phi(t,p)\,z)\,, \end{array}$$

where $\phi(t,p)\,z = v(t,p,z)$, the mild solution for the linear problem given by p with initial condition given by z.

The operators $\phi(t,p) : C(\bar{U}) \to C(\bar{U})$ satisfy:

(i) Linear semicocycle property:

$$\phi(t+s,p) = \phi(t,p{\cdot}s)\,\phi(s,p)\,, \quad p \in P, \ t,s \geq 0;$$

(ii) For $p \in P$ and $t > 0$, they are compact and strongly positive: if $z > 0$, then $\phi(t,p)\,z \gg 0$.

Remark: for $n-$dimensional systems, additional conditions are needed.

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Remark: for $n-$dimensional systems, additional conditions are needed.

A key property: the linear semiflow admits a continuous separation

This result comes as a step by step generalization of a previous result:

(i) The Perron-Frobenius theorem (1907–1909): finite-dimensional case, i.e., nonnegative and irreducible $n \times n$ matrices;

(ii) The Krein-Rutman theorem (1950): infinite-dimensional case, $A : X \to X$ compact and strongly positive linear operator on the strongly ordered Banach space X;

(iii) Poláčik and Tereščák (1993): the result for vector bundle maps on $P \times X$;

(iv) Shen and Yi (1998): the result for linear skew-product semiflows on $P \times X$.

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The linear semiflow admits a continuous separation

There are two families of subspaces $\{X_1(p)\}_{p\in P}$ and $\{X_2(p)\}_{p\in P}$ of $C(\bar{U})$ which satisfy:

(1) $C(\bar{U}) = X_1(p) \oplus X_2(p)$, with a continuous variation in P;

(2) $X_1(p) = \langle e(p)\rangle$, with $e(p) \gg 0$ and $\|e(p)\| = 1$ for any $p \in P$;

(3) $X_2(p) \cap C(\bar{U})_+ = \{0\}$ for any $p \in P$;

(4) for any $t > 0$, $p \in P$,

$$\begin{aligned} \phi(t,p)X_1(p) &= X_1(p{\cdot}t)\,, \\ \phi(t,p)X_2(p) &\subseteq X_2(p{\cdot}t)\,; \end{aligned}$$

(5) there are $M > 0$, $\delta > 0$ such that for any $p \in P$, $z \in X_2(p)$ with $\|z\| = 1$ and $t > 0$,

$$\|\phi(t,p)\,z\| \le M\,e^{-\delta t}\|\phi(t,p)\,e(p)\|\,.$$

Principal bundle and principal spectrum (Mierczyński and Shen, 2004)

- The *principal bundle* is the 1-dim invariant subbundle

$$\bigcup_{p\in P} \{p\} \times X_1(p)\,.$$

- The *principal spectrum* Σ_{pr} is the Sacker-Sell spectrum or continuous spectrum of the restriction of τ_L to the principal bundle. In general $\Sigma_{\text{pr}} = [\alpha_P, \lambda_P]$.
- When P is uniquely ergodic, $\Sigma_{\text{pr}} = \{\lambda_P\}$, for the upper Lyapunov exponent λ_P.

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The associated 1-dimensional linear cocycle $c(t, p)$

To each linear coefficient $h \in C(P \times \bar{U})$ we associate a 1-dim linear semicocycle $c(t, p)$, the one driving the dynamics of τ_L when restricted to the principal bundle, i.e., $c(t, p)$ is the positive number such that

$$\phi(t, p)\, e(p) = c(t, p)\, e(p{\cdot}t)\,, \quad t \geq 0\,,\; p \in P\,.$$

$c(t, p)$ can be extended to a linear cocycle

$$c(t + s, p) = c(t, p{\cdot}s)\, c(s, p)\,, \quad p \in P,\; t, s \in \mathbb{R}\,,$$

by taking $c(-t, p) = 1/c(t, p{\cdot}(-t))$ for any $t > 0$ and $p \in P$.

Besides,

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The case of null upper Lyapunov exponent: $\lambda_P = 0$

Let $C_0(P \times \bar{U}) = \{h \in C(P \times \bar{U}) \mid \lambda_P(h) = 0\}$.
Theorem: It is a complete metric space.

We classify the maps $h \in C_0(P \times \bar{U})$ depending on whether the associated 1-dim cocycle $c(t, p)$ is "bounded" or not:

$$B(P \times \bar{U}) = \{h \in C_0(P \times \bar{U}) \mid \sup_{t \in \mathbb{R}} |\ln c(t, p)| < \infty \text{ for any } p \in P\},$$

$$\mathcal{U}(P \times \bar{U}) = C_0(P \times \bar{U}) \setminus B(P \times \bar{U}).$$

Remark: if the flow on P is periodic, then $C_0(P \times \bar{U}) = B(P \times \bar{U})$.

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Theorem (in line with the classical result by Gottschalk and Hedlund (1955) for maps in $C_0(P) = \{a \in C(P) \mid \int_P a\, d\nu = 0\}$). The following conditions are equivalent:

(i) For any $p \in P$, $\sup_{t\in\mathbb{R}} |\ln c(t,p)| < \infty$.

(ii) There exists a $p_0 \in P$ such that $\sup_{t\in\mathbb{R}} |\ln c(t,p_0)| < \infty$.

(iii) There exists a $p_0 \in P$ such that

$$\text{either} \quad \sup_{t\geq 0} |\ln c(t,p_0)| < \infty \quad \text{or} \quad \sup_{t\leq 0} |\ln c(t,p_0)| < \infty\,.$$

(iv) There exists a function $k \in C(P)$ such that

$$k(p{\cdot}t) - k(p) = \ln c(t,p) \ \text{ for all } \ p \in P,\ t \in \mathbb{R}\,.$$

Some results for general positive linear 1-dim cocycles-II

Theorem (in line with the oscillation result by Johnson (1978) for maps in $C_0(P)$ with unbounded primitive).

If the associated 1-dim linear cocycle $c(t,p)$ does not satisfy the previous conditions, then there exists an invariant and residual set $P_{\rm o} \subset P$ such that for any $p \in P_{\rm o}$,

$$\liminf_{t\to\pm\infty} c(t,p) = 0\,,$$

$$\limsup_{t\to\pm\infty} c(t,p) = \infty\,.$$

Dynamics of the linear semiflow when $\lambda_P = 0$

If $h \in B(P \times \bar{U})$:

(i) For $z > 0$, bounded orbits both above and away from 0.

(ii) There is a strongly positive continuous equilibrium $\widehat{e} : P \to \operatorname{Int} C(\bar{U})_+$, $\widehat{e}(p{\cdot}t) = \phi(t,p)\,\widehat{e}(p)$ for $p \in P$, $t \geq 0$ (thus, infinitely many).

If $h \in \mathcal{U}(P \times \bar{U})$:

(i) There is an invariant and residual set $P_{\text{o}} \subset P$ such that for any $p \in P_{\text{o}}$ and any $z > 0$, the orbit $\phi(t,p)\,z$ has a strong oscillating behaviour:

$$\liminf_{t\to\infty} \|\phi(t,p)\,z\| = 0 \quad \text{and} \quad \limsup_{t\to\infty} \|\phi(t,p)\,z\| = \infty\,.$$

(ii) There is a pinched invariant compact set in $P \times (\operatorname{Int} C(\bar{U})_+ \cup \{0\})$.

$\mathcal{U}(P \times \bar{U})$ is a residual set in $C_0(P \times \bar{U})$.

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Back to the linear-dissipative problems

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With $h \in C(P \times \bar{U})$ and $g : P \times \bar{U} \times \mathbb{R} \to \mathbb{R}$ continuous and of class C^1 with respect to y, arguing as in the linear case, one can build the associated skew-product semiflow induced by mild solutions. In principle it is only locally defined.

$$\begin{array}{rccc} \tau : & \mathbb{R}_+ \times P \times C(\bar{U}) & \longrightarrow & P \times C(\bar{U}) \\ & (t, p, z) & \mapsto & (p{\cdot}t, u(t, p, z)) \end{array}$$

Assumptions

We assume that $h \in C_0(P \times \bar{U})$ and for the non-linear term g:

(c1) $g(p,x,0) = \dfrac{\partial g}{\partial y}(p,x,0) = 0$ for any $p \in P$ and $x \in \bar{U}$;

(c2) $y\, g(p,x,y) \le 0$ for any $p \in P$, $x \in \bar{U}$ and $y \in \mathbb{R}$;

(c3) $\displaystyle\lim_{|y|\to\infty} \frac{g(p,x,y)}{y} = -\infty$ uniformly on $P \times \bar{U}$;

(c4) $g(p,x,-y) = -g(p,x,y)$ for any $p \in P$, $x \in \bar{U}$ and $y \in \mathbb{R}$;

(c5) there exists an $r_0 > 0$ such that $g(p,x,y) = 0$ if and only if $|y| \le r_0$.

The skew-product semiflow is globally defined and strongly monotone.

With a general $h \in C(P \times \bar{U})$ and conditions (c1)-(c3) Cardoso, Langa and Obaya (2016) prove the existence of a compact absorbing set, so that there is a global attractor $\mathbb{A} \subset P \times C(\bar{U})$.

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With a general $h \in C(P \times \bar{U})$ and conditions (c1)-(c3) Cardoso, Langa and Obaya (2016) prove the existence of a compact absorbing set, so that there is a global attractor $\mathbb{A} \subset P \times C(\bar{U})$.

The global attractor $\mathbb{A}$ and the cocycle attractor

- $\mathbb{A} \subset P \times C(\bar{U})$ is a compact set;
- $\mathbb{A}$ is invariant: $\tau_t(\mathbb{A}) = \mathbb{A}$ for any $t \geq 0$;
- $\mathbb{A}$ (forwards) attracts bounded sets $\mathbb{B} \subset P \times C(\bar{U})$: $\lim_{t\to\infty} \mathrm{dist}(\tau_t(\mathbb{B}), \mathbb{A}) = 0$ for the Hausdorff distance.

Since P is compact and $\mathbb{A}$ is the global attractor, the non-autonomous set $\{A(p)\}_{p\in P}$ given by

$$A(p) = \{z \in C(\bar{U}) \mid (p, z) \in \mathbb{A}\}$$

is the cocycle attractor:

- it is compact: every $A(p)$ is compact in $C(\bar{U})$;
- it is invariant: for $p \in P$, $u(t, p, A(p)) = A(p{\cdot}t)$ for $t \geq 0$;
- it pullback attracts all bounded subsets $B \subset C(\bar{U})$, that is,

$$\lim_{t\to\infty} \mathrm{dist}(u(t, p{\cdot}(-t), B), A(p)) = 0 \quad \text{for any } p \in P\,.$$

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Global and cocycle attractors

Taking

$$a(p) = \inf A(p) \quad \text{and} \quad b(p) = \sup A(p) \quad \text{for any } p \in P,$$

$a(p)$ and $b(p)$ are semicontinuous equilibria for τ and

$$\mathbb{A} \subseteq \bigcup_{p \in P} \{p\} \times [a(p), b(p)]\,.$$

With condition (c4), $a(p) = -b(p)$ and we just study $b(p)$.

On the structure of the attractor $\mathbb{A}$

The study heavily relies upon the dynamical study of the linear problems.

Theorem: Basically, there are two situations:

- If $h \in B(P \times \bar{U})$, $\mathbb{A}$ is a wide set: there is an $r_* > 0$ such that

$$A(p) = \{r\,\widehat{e}(p) \mid |r| \leq r_*\} \subset X_1(p) \quad \text{for any } \; p \in P,$$

  for a strongly positive continuous equilibrium $\widehat{e} : P \to C(\bar{U})_+$ of the linear problem.

- If $h \in \mathcal{U}(P \times \bar{U})$, $\mathbb{A}$ is a pinched set with a complex dynamical structure. In some cases, the attractor is chaotic.

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The attractor when $h \in \mathcal{U}(P \times \bar{U})$

Theorem:

(i) There exists an invariant residual set $P_s \subsetneq P$ such that $b(p) = 0$ for any $p \in P_s$.

(ii) The set $P_f = P \setminus P_s$ is an invariant dense set of first category and $b(p) \gg 0$ for any $p \in P_f$.

How complex can the dynamics inside $\mathbb{A}$ be?

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Chaotic dynamics when $\nu(P_{\mathrm{f}}) = 1$

Theorem: The global attractor $\mathbb{A}$ is *fiber-chaotic in measure in the sense of Li-Yorke*, that is, there exists a set $P_{\mathrm{ch}} \subset P$ of full measure such that for every $p \in P_{\mathrm{ch}}$, every pair $z_1, z_2 \in A(p)$ ($z_1 \neq z_2$) is a (fiber) Li-Yorke pair:

$$\liminf_{t\to\infty} \|u(t,p,z_2) - u(t,p,z_1)\| = 0\,,$$
$$\limsup_{t\to\infty} \|u(t,p,z_2) - u(t,p,z_1)\| > 0\,.$$

Discontinuous non-autonomous bifurcation results

One-parametric family ($\gamma \in \mathbb{R}$) of scalar reaction-diffusion problems over a minimal, uniquely ergodic and aperiodic flow $(P, \cdot, \mathbb{R})$, with Neumann or Robin boundary conditions, given for each $p \in P$ by

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta\, y + (\gamma + h(p{\cdot}t, x))\, y + g(p{\cdot}t, x, y)\,, \quad t > 0\,,\ x \in U, \\ \alpha(x)\, y + \dfrac{\partial y}{\partial n} = 0\,, \quad t > 0\,,\ x \in \partial U, \end{cases}$$

where $h \in \mathcal{U}(P \times \bar{U})$ and $g : P \times \bar{U} \times \mathbb{R} \to \mathbb{R}$ is continuous, of class C^1 with respect to y, it satisfies conditions (c1)-(c5) and also

(c6) $g(p, x, y)$ is convex in y for $y \leq 0$ and concave in y for $y \geq 0$.

For instance, g might be the map

$$g(p,x,y) = \begin{cases} k(p,x)\,(y+r_0)^3\,, & y \le -r_0 \\ 0\,, & -r_0 \le y \le r_0 \\ -k(p,x)\,(y-r_0)^3\,, & y \ge r_0 \end{cases}$$

for a positive map $k \in C(P \times \bar{U})$ and the constant r_0 in (c5).

CHAFEE, INFANTE, A bifurcation problem for a nonlinear partial differential equation of parabolic type, *Applicable Anal.* **4** (1974).

Some non-autonomous versions of this equation together with bifurcation problems: CARVALHO, LANGA, ROBINSON, Structure and bifurcation of pullback attractors in a non-autonomous Chafee-Infante equation, *Proc. Amer. Math. Soc.* **140** (2012).

Notation: $\mathbb{A}_\gamma$ is the global attractor of the skew-product semiflow τ_γ for the value $\gamma \in \mathbb{R}$ and b_γ is its upper boundary map.

Theorem:

(i) If $\gamma < 0$, then $\mathbb{A}_\gamma = P \times \{0\}$ is the global attractor and it is globally exponentially stable.

(ii) If $\gamma = 0$, then the global attractor $\mathbb{A}_0 \subseteq \bigcup_{p \in P} \{p\} \times [-b_0(p), b_0(p)]$ is a pinched set which contains a unique minimal set $P \times \{0\}$. If $\nu(P_{\mathrm{f}}) = 1$, then $\mathbb{A}_0$ is fiber-chaotic in measure in the sense of Li-Yorke.

(iii) If $\gamma > 0$, then the global attractor $\mathbb{A}_\gamma \subseteq \bigcup_{p \in P} \{p\} \times [-b_\gamma(p), b_\gamma(p)]$ with $b_\gamma(p) \gg 0$ for every $p \in P$ and the maps $\pm b_\gamma$ define continuous equilibria. The copies of the base $K_\gamma^\pm = \{(p, \pm b_\gamma(p)) \mid p \in P\}$ are globally exponentially stable minimal sets in $P \times \operatorname{Int} X_\pm$, whereas the minimal set $P \times \{0\}$ is unstable.

The cases $\gamma < 0$ and $\gamma > 0$: in Cardoso, Langa and Obaya (2016).

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THANK YOU FOR YOUR ATTENTION :)