Revisiting slow fast dynamics with piecewise linear differential systems.

A.E. Teruel

Sesión I: Piecewise-smooth systems, averaging and hybrid systems.

03/10/2018

Carmona, Fernández, Desroches, A.E.T. *Saddle-node canard orbits in planar PWL system* Work in progress.

Carmona, Fernández, Desroches, A.E.T. *Transitory canards and maximal canard orbits in planar PWL system* Work in progress.

Desroches, Guillamon, Ponce, Prohens, Rodrigues, A.E.T. *Folded nodes and mixed-mode oscillations in piecewise-linear slow-fast systems* SIAM Rev. 2016

Fernández, Desroches, Krupa, A.E.T.*Canard solutions in planar piecewise linear systems with three zones* Dyn. Sys. 2016.

Prohens, A.E.T. *Canard trajectories in 3D piecewise linear systems* DCDS, 2013

Universitat de les Illes Balears | IAC3 Institute of Applied Computing & Community Code.

Ministerio de Economía, Industria y Competitividad

European Union European Regional Development Fund "A way to make Europe"

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- Canard phenomenon. Supercritical Hopf bifurcation
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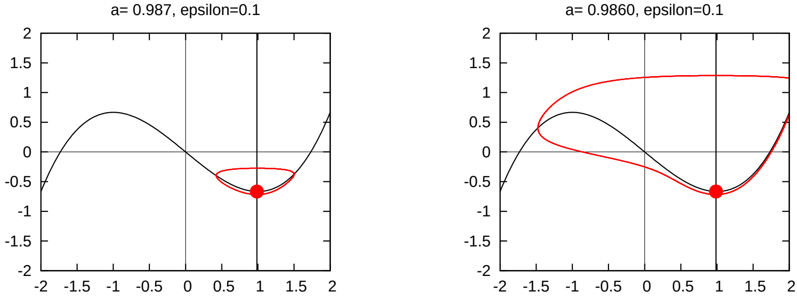
- Subcritical Hopf bifurcation
- PWL Saddle-node canard orbits
- Conclusions

Canard orbit, Canard explosion

- Limit cycle flowing close to a repelling invariant manifold.
- Sudden growing of the amplitude of the canard limit cycles.
- Example: Van der Pol system

$$\left.\begin{aligned}\dot{x} &= y - f(x)\\ \dot{y} &= \varepsilon(a - x)\end{aligned}\right\}, \quad f(x) = x\left(\frac{x^2}{3} - 1\right), \quad 0 < \varepsilon \ll 1$$



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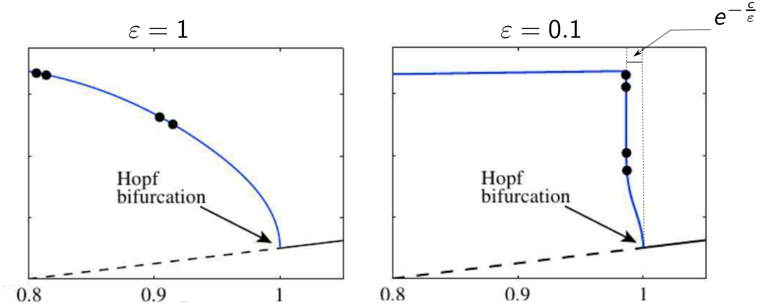
$$\left.\begin{aligned}\dot{x} &= y - f(x)\\ \dot{y} &= \varepsilon(a - x)\end{aligned}\right\},\quad f(x) = x\left(\frac{x^2}{3} - 1\right),\quad J = \begin{pmatrix} 1 - a^2 & 1 \\ -\varepsilon & 0 \end{pmatrix}$$

$$\triangle = (1 - a^2)^2 - 4\varepsilon$$



Canard explosion (geometrical approach)

Change of time $\tau = \varepsilon t$

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Considering the limit problems, $\varepsilon = 0$

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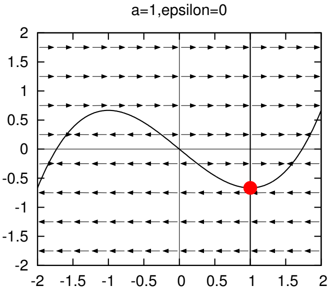

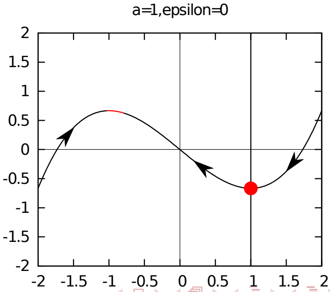

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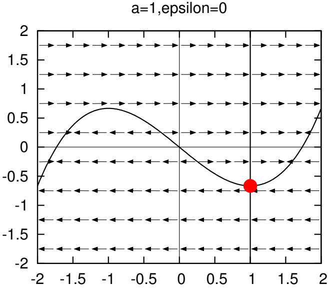

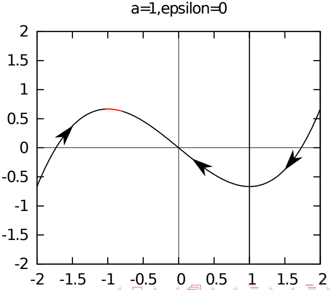

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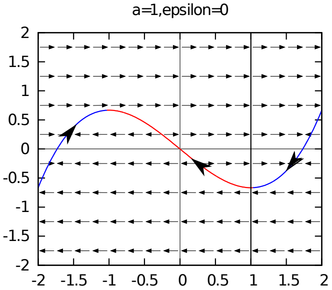

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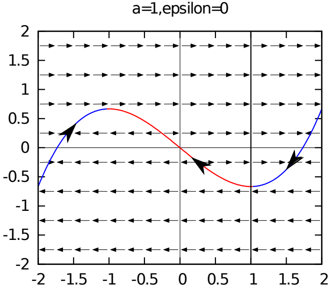

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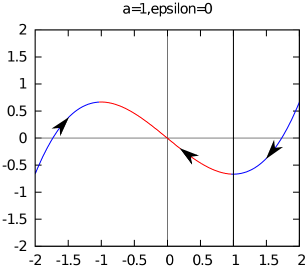

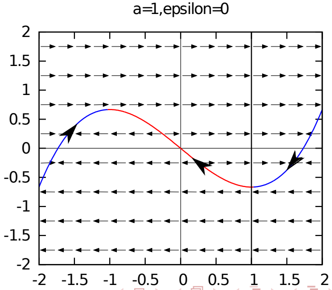

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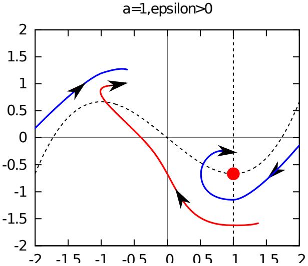

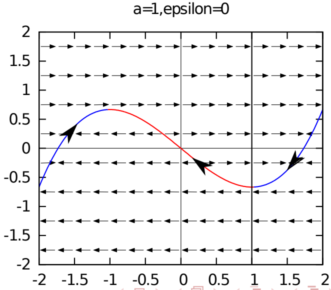

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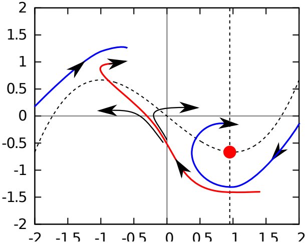

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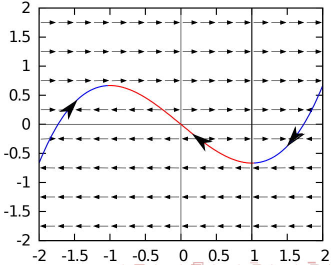

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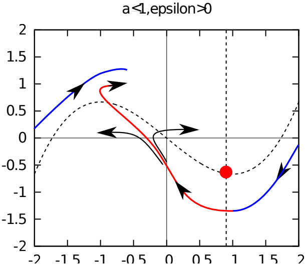

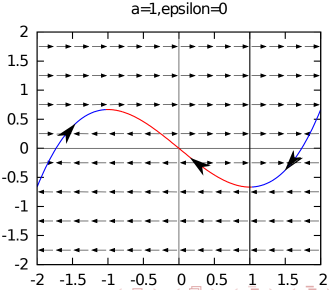

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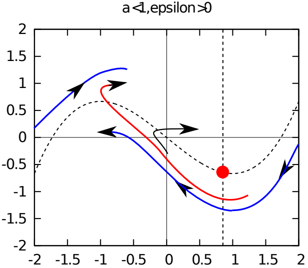

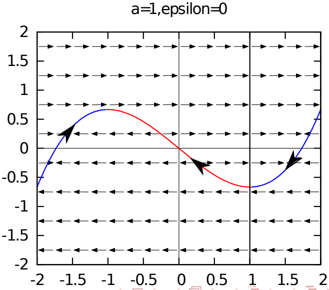

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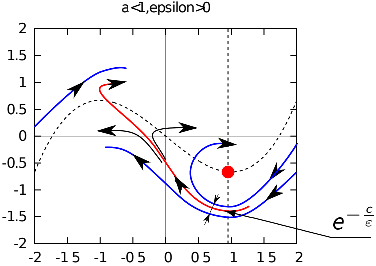

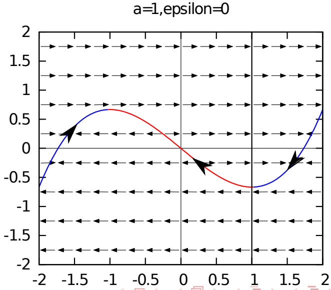

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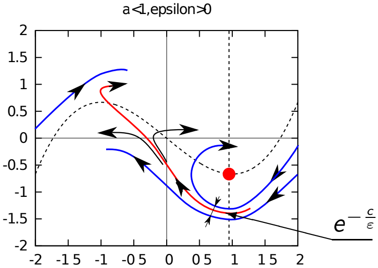

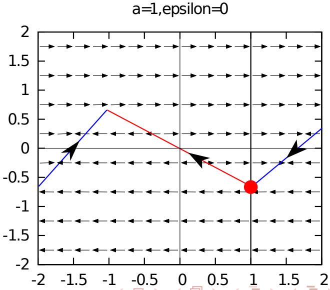

PWL Canards

- M. Komuro and T. Saito *"Lost solution" in a piecewise linear system*, IEICE Trans., vol. 74 (1991), pp. 3625–3627.

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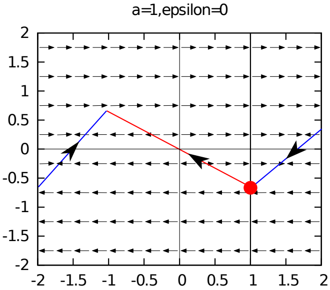

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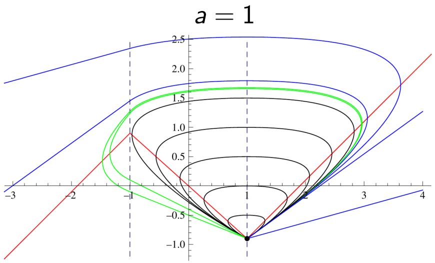

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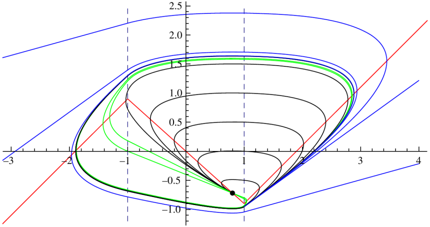

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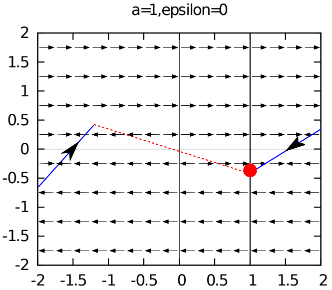

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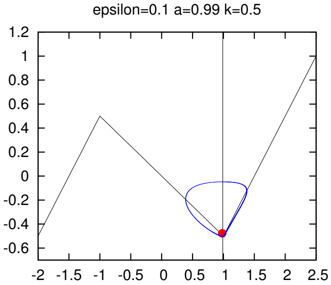

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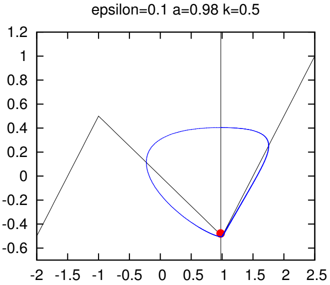

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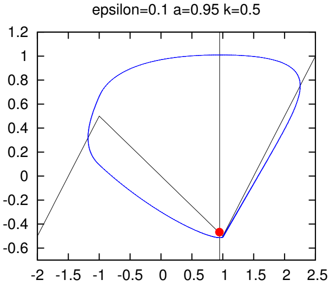

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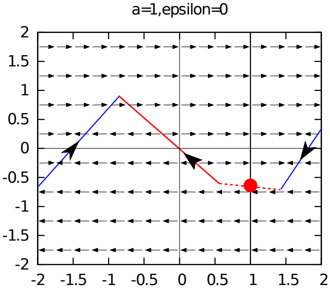
fixed values of k, m, b and ε.



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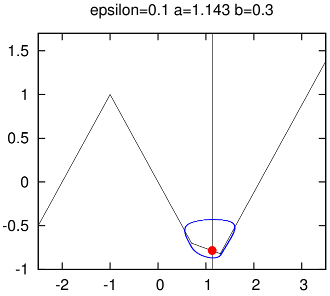


PWL Canards

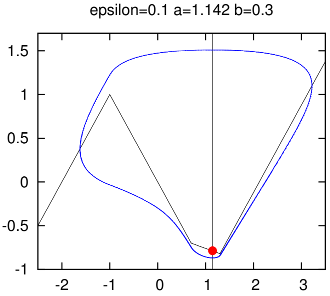
- M. Komuro and T. Saito *"Lost solution" in a piecewise linear system*, IEICE Trans., vol. 74 (1991), pp. 3625–3627.
- N. Arima, H. Okazaki, and H. Nakano *A generation mechanism of canards in a piecewise linear system*, IEICE Trans., vol. 80 (1997), pp. 447–453.
- M. Desroches, E. Freire, S.J. Hogan, E. Ponce and P. Thota *Canards in PWL systems: explosions and super-explosions*, Proc. Royal S. 2013.

$$\left.\begin{array}{l}\dot{x} = y - f(x)\\ \dot{y} = \varepsilon(a-x)\end{array}\right\}$$

$$\begin{cases} x+k+1 & x<-1\\ -kx & x\in(-1,1-b)\\ mx-(m+k)(1-b) & |x-1|<b\\ -x+2bm-k(1-b) & x>1+b\end{cases}$$

$$\triangle = m^2 - 4\varepsilon < 0$$

fixed values of k, m, b and ε.



PWL Canards

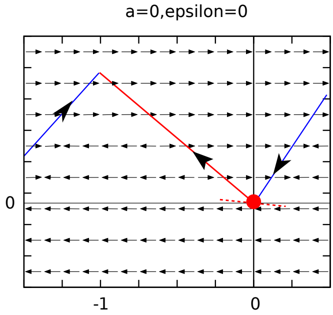
- S. Fernández-García, M. Desroches, M. Krupa and A.E. Teruel *Canard solutions in planar piecewise linear systems with 3 zones*, Dyn.Sys 2015

$$\left.\begin{array}{l}\dot{x} = y - f(x)\\ \dot{y} = \varepsilon(a - x)\end{array}\right\}$$

$$\left\{\begin{array}{ll} x + k + 1 - (k+m)\sqrt{\varepsilon} - ma & x < -1\\ -kx - (k+m)\sqrt{\varepsilon} - ma & x \in (-1, -\sqrt{\varepsilon})\\ m(x-a) & |x| < \sqrt{\varepsilon}\\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a) & x > \sqrt{\varepsilon}\end{array}\right.$$

$$\triangle = m^2 - 4\varepsilon < 0$$

$$k = 1, |m| < 2\sqrt{\varepsilon}$$



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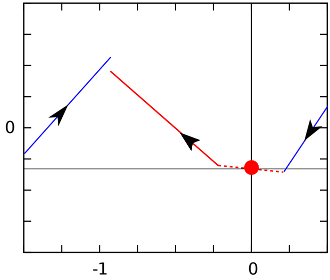
$$\triangle = m^2 - 4\varepsilon < 0$$

$$k = 1, |m| < 2\sqrt{\varepsilon}$$

a=0,epsilon>0



PWL Canards

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$$\left.\begin{array}{l}\dot{x} = y - f(x)\\ \dot{y} = \varepsilon(a - x)\end{array}\right\}$$

$$\begin{cases} -kx - (k+m)\sqrt{\varepsilon} - ma & x < -\sqrt{\varepsilon} \\ m(x-a) & |x| < \sqrt{\varepsilon} \\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a) & x > \sqrt{\varepsilon} \end{cases}$$

$$\triangle = m^2 - 4\varepsilon < 0$$

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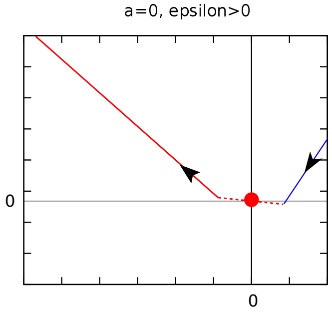

PWL Canards

- S. Fernández-García, M. Desroches, M. Krupa and A.E. Teruel *Canard solutions in planar piecewise linear systems with 3 zones*, Dyn.Sys 2015
- Slow manifolds connection at $a = \tilde{a}(\varepsilon, m)$

$$\left.\begin{aligned} \dot{x} &= y - f(x) \\ \dot{y} &= \varepsilon(a - x) \end{aligned}\right\}$$

$$\begin{cases} -kx - (k+m)\sqrt{\varepsilon} - ma & x < -\sqrt{\varepsilon} \\ m(x-a) & |x| < \sqrt{\varepsilon} \\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a) & x > \sqrt{\varepsilon} \end{cases}$$

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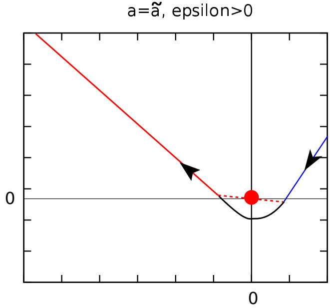


PWL Canards

- S. Fernández-García, M. Desroches, M. Krupa and A.E. Teruel *Canard solutions in planar piecewise linear systems with 3 zones*, Dyn.Sys 2015
- Slow manifolds connection at $a = \tilde{a}(\varepsilon, m)$
- Clossing equations of canard orbits $\Gamma(h, a, m, \varepsilon)$ through $(-\sqrt{\varepsilon}, h)$

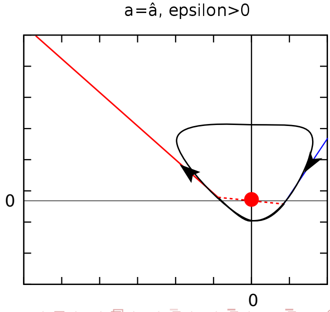
$$\begin{pmatrix} F(\tau, a, m, \varepsilon) \\ G(\tau, a, m, \varepsilon) \end{pmatrix} + \begin{pmatrix} \xi_1(\tau, a, m, \varepsilon) \\ \xi_2(\tau, a, m, \varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\left.\begin{array}{l} \dot{x} = y - f(x) \\ \dot{y} = \varepsilon(a - x) \end{array}\right\}$$

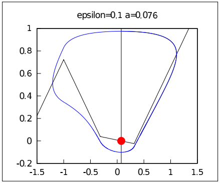
$$\begin{cases} -kx - (k+m)\sqrt{\varepsilon} - ma & x < -\sqrt{\varepsilon} \\ m(x - a) & |x| < \sqrt{\varepsilon} \\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a) & x > \sqrt{\varepsilon} \end{cases}$$

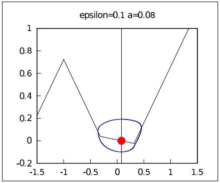
$$\triangle = m^2 - 4\varepsilon < 0$$

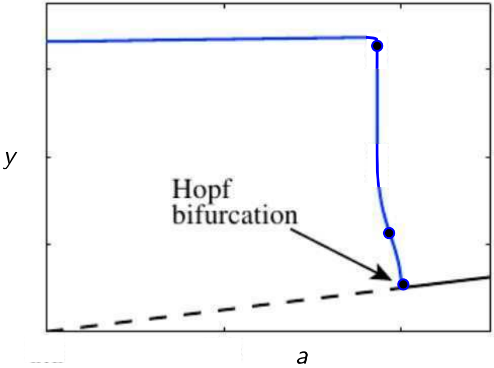
$$k = 1, |m| < 2\sqrt{\varepsilon}$$

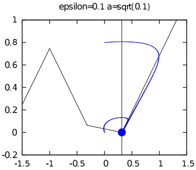


PWL Canard Explosion







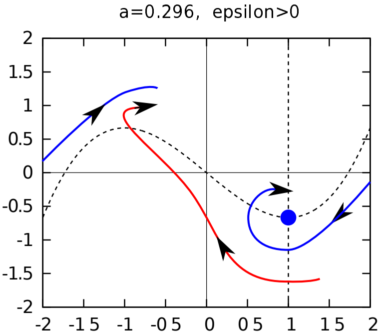


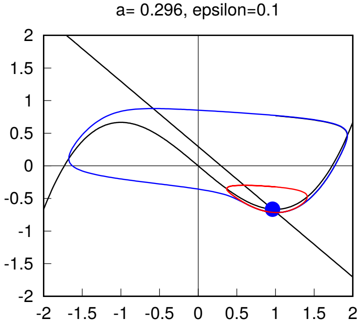
Subcritical Hopf bifurcation

- Example: FitzHugh-Nagumo

$$\left.\begin{aligned}\dot{x} &= y - f(x)\\ \dot{y} &= \varepsilon(a - x - y)\end{aligned}\right\}, \quad f(x) = x\left(\frac{x^2}{3} - 1\right), \qquad J = \begin{pmatrix} 1 - (3a)^{\frac{1}{3}} & 1 \\ -\varepsilon & -\varepsilon \end{pmatrix}$$

$$\triangle = (3a)^{\frac{1}{3}}\varepsilon > 0$$





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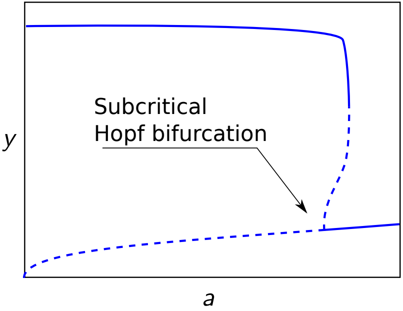
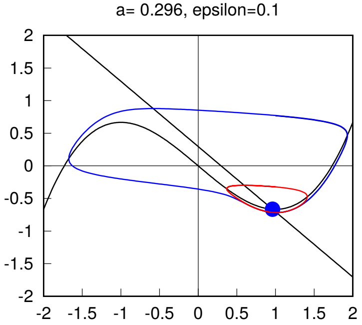


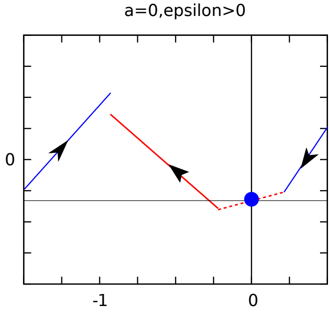


PWL SN-canard

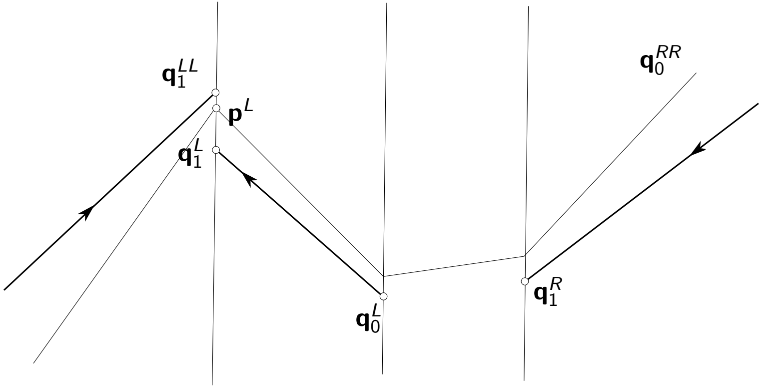
$$\left.\begin{aligned}\dot{x} &= y - f(x)\\ \dot{y} &= \varepsilon(a - x)\end{aligned}\right\}$$

$$\begin{cases} x + k + 1 - (k+m)\sqrt{\varepsilon} - ma & x < -1\\ -kx - (k+m)\sqrt{\varepsilon} - ma & x < -\sqrt{\varepsilon}\\ m(x-a) & |x| < \sqrt{\varepsilon}\\ x - \sqrt{\varepsilon} + m(\sqrt{\varepsilon} - a) & x > \sqrt{\varepsilon}\end{cases}$$

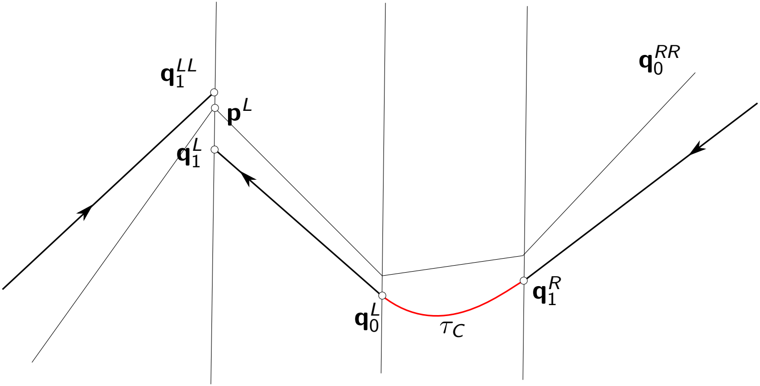
$$\triangle = m^2 - 4\varepsilon < 0, \quad m = \sqrt{\varepsilon} > 0$$



PWL SN-canard



PWL SN-canard

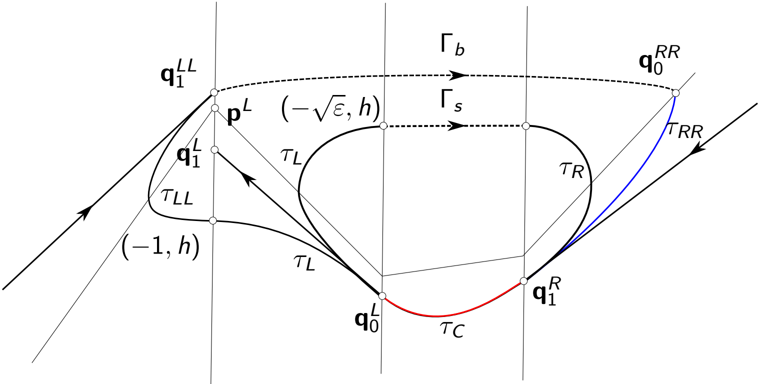


Th. *Slow manifolds connection*

$$\tilde{a}(k,\sqrt{\varepsilon}) = -\frac{e^{\frac{\pi}{\sqrt{3}}}-1}{e^{\frac{\pi}{\sqrt{3}}}+1}\sqrt{\varepsilon} - \frac{e^{\frac{\pi}{\sqrt{3}}}}{\left(e^{\frac{\pi}{\sqrt{3}}}+1\right)^2}\left(\frac{1-k^2}{k^2}\right)\varepsilon^{3/2} + O(\varepsilon^2),$$

$$\tau_C(k,\sqrt{\varepsilon}) = \frac{2\pi}{\sqrt{3}}\frac{1}{\sqrt{\varepsilon}} - \frac{1+k}{k} + \frac{1-k^2}{2k^2}\sqrt{\varepsilon} + O(\varepsilon^2).$$

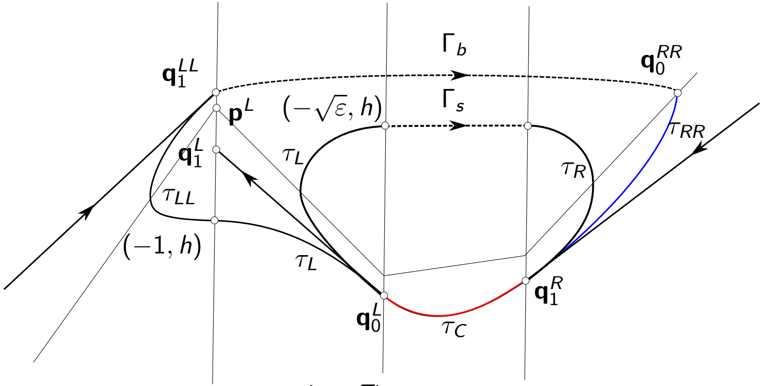
PWL SN-canard



Th. *Fixed* $\varepsilon \ll 1$, *for every* $0 < h < h_{\mathbf{p}_L}$ *there exists* $\hat{a}(k, \sqrt{\varepsilon})$ *with the same first terms of the Taylor series expansion as* $\tilde{a}(k, \sqrt{\varepsilon})$ *such that the system exhibits a canard orbit* $\Gamma_s(h, \sqrt{\varepsilon})$ *or* $\Gamma_b(h, \sqrt{\varepsilon})$.

$$\tilde{a}(k, \sqrt{\varepsilon}) = -\frac{e^{\frac{\pi}{\sqrt{3}}} - 1}{e^{\frac{\pi}{\sqrt{3}}} + 1}\sqrt{\varepsilon} - \frac{e^{\frac{\pi}{\sqrt{3}}}}{\left(e^{\frac{\pi}{\sqrt{3}}} + 1\right)^2}\left(\frac{1 - k^2}{k^2}\right)\varepsilon^{3/2} + O(\varepsilon^2),$$

PWL SN-canard

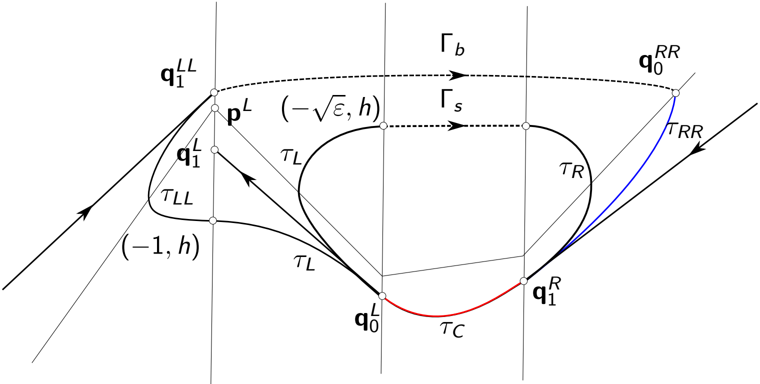


- Chosen a canard orbit $\Gamma_s(h, \sqrt{\varepsilon})$.

$$\tau_L = \frac{1}{\lambda_L^s} \ln\left(1 + \frac{h + (\sqrt{\varepsilon} + \lambda_L^s)(\sqrt{\varepsilon} + a)}{(\lambda_L^q - \lambda_L^s)(\sqrt{\varepsilon} + a)}\right),$$

$$\tau_R = -\frac{1}{\lambda_R^s} \ln\left(1 + \frac{(\sqrt{\varepsilon} + \lambda_R^s)(\sqrt{\varepsilon} - a) - h}{(\lambda_R^q - \lambda_R^s)(\sqrt{\varepsilon} - a)}\right),$$

PWL SN-canard



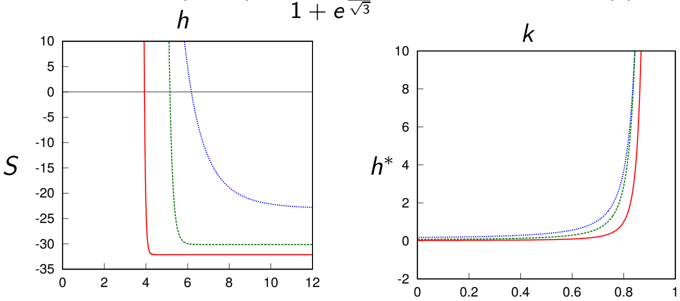
- Condition of non-hyperbolicity on $\Gamma_s(h, \sqrt{\varepsilon}) \to e^{t_L\tau_L + t_C\tau_C + t_R\tau_R} = 1.$

$$S(h, \sqrt{\varepsilon}) = \left(1 + \frac{h + (\sqrt{\varepsilon} + \lambda_L^s)(\sqrt{\varepsilon} + a)}{(\lambda_L^q - \lambda_L^s)(\sqrt{\varepsilon} + a)}\right)^{\frac{k}{\lambda_L^s}} \left(1 + \frac{(\sqrt{\varepsilon} + \lambda_R^s)(\sqrt{\varepsilon} - a) - h}{(\lambda_R^q - \lambda_R^s)(\sqrt{\varepsilon} - a)}\right)^{\frac{1}{\lambda_R^s}} - e^{\sqrt{\varepsilon}\tau_C}.$$

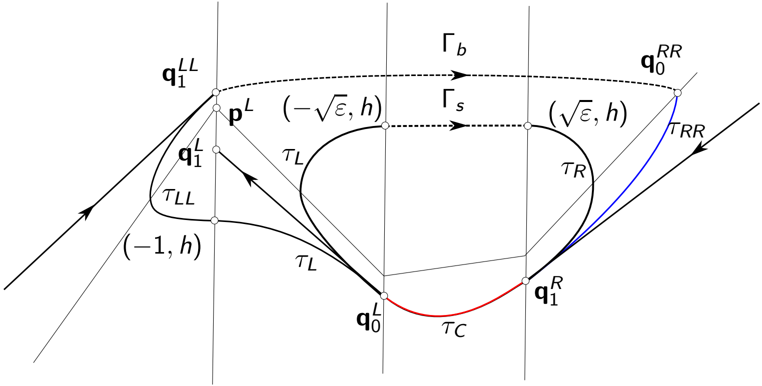
PWL SN-canard

- **Th.** *Assume* $a = \tilde{a}(h, \sqrt{\varepsilon})$ *and* $\varepsilon \ll 1$.
  - *If* $k \geq 1$, *then* $S(h, \sqrt{\varepsilon}) \neq 0$.
  - *If* $k < 1$ *and* $0 < h < h_{\mathbf{p}_L}$ *is a simple of* $S(h, \sqrt{\varepsilon})$, *then* $\Gamma_s(h, \sqrt{\varepsilon})$ *is a non-hyperbolic canard orbit.*
- **Th.** *Assume* $a = \tilde{a}(h, \sqrt{\varepsilon})$, $\varepsilon \ll 1$, *and* $k < 1$.
  - a) $\lim_{h \searrow 0} S(0, \sqrt{\varepsilon}) > 0$ *and* $\lim_{h \nearrow +\infty} S(h, \sqrt{\varepsilon}) = -e^{\frac{2\pi}{\sqrt{3}}}$.
  - b) *Let* $h^* > 0$ *be a solution of* $S(h, \sqrt{\varepsilon}) = 0$, *then* $\left.\frac{\partial S}{\partial h}\right|_{(h^*, \sqrt{\varepsilon})} < 0$.
  - c) *Let* $h^*(k, \sqrt{\varepsilon})$ *be the positive solution of* $S(h, \sqrt{\varepsilon}) = 0$, *then*

$$h^*(k, \sqrt{\varepsilon}) = \frac{2}{1 + e^{\frac{\pi}{\sqrt{3}}}} k^{\frac{k^2}{k^2-1}} e^{\frac{\pi}{\sqrt{3}} \frac{1-2\varepsilon}{1-k^2}} \sqrt{\varepsilon} + O(\varepsilon)$$



PWL SN-canard

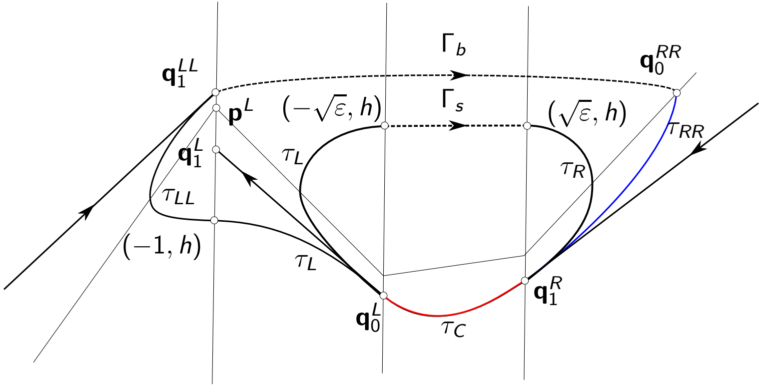


- Chosen a canard orbit $\Gamma_b(h, \sqrt{\varepsilon})$.

$$\tilde{\tau}_L = \frac{1}{\lambda_L^s} \ln(1 + \dots) \quad \tau_{LL} = -\frac{1}{\lambda_{LL}^s} \ln(1 + \dots),$$

$$\tau_{RR} = -\frac{1}{\lambda_R^s} \ln(1 + \dots).$$

PWL SN-canard



- Condition of non-hyperbolicity on $\Gamma_s(h, \sqrt{\varepsilon})$

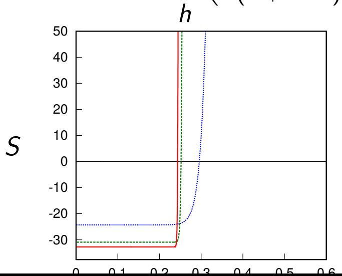
$$e^{t_{LL}\tau_{LL}+t_L\tau_L+t_C\tau_C+t_R\tau_R} = 1.$$

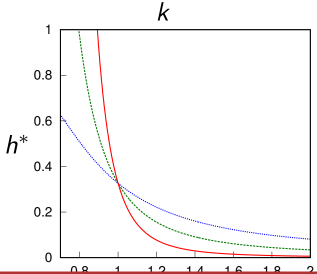
$$S(h, \sqrt{\varepsilon}) = (1+\dots)^{\frac{k}{\lambda_L^s}}\,(1+\dots)^{\frac{1}{\lambda_{LL}^s}}\,(1+\dots)^{\frac{1}{\lambda_R^s}} - e^{\sqrt{\varepsilon}\tau_C}.$$

PWL SN-canard

- **Th.** *Assume* $a = \tilde{a}(h, \sqrt{\varepsilon})$ *and* $\varepsilon \ll 1$.
  - *If* $k < 1$, *then* $S(h, \sqrt{\varepsilon}) \neq 0$.
  - *If* $k \geq 1$, *exists* $0 < h_M < h_{\mathbf{p}_L}$ *s. t. if* $0 < h < h_M$ *is a zero simple of* $S(h, \sqrt{\varepsilon})$, *then* $\Gamma_b(h, \sqrt{\varepsilon})$ *is a non-hyperbolic canard orbit.*
- **Th.** *Assume* $a = \tilde{a}(h, \sqrt{\varepsilon})$, $\varepsilon \ll 1$, *and* $1 < k < (1 + e^{\pi/2})$.
  - a) $S(0, \sqrt{\varepsilon}) < 0$ *and* $S(k, \sqrt{\varepsilon}) > 0$.
  - b) *Let* $0 < h^* < k$ *be a solution of* $S(h, \sqrt{\varepsilon}) = 0$, *then* $\left.\frac{\partial S}{\partial h}\right|_{(h^*, \sqrt{\varepsilon})} > 0$.
  - c) *Let* $h^*(k, \sqrt{\varepsilon})$ *be the positive solution of* $S(h, \sqrt{\varepsilon}) = 0$, *then*

$$h^*(k, \sqrt{\varepsilon}) = \left(\frac{2}{k(1 + e^{\frac{\pi}{\sqrt{\varepsilon}}})}\right)^{\frac{k^2-1}{k^2}} \left(\frac{1+k}{e^{\frac{\pi}{\sqrt{\varepsilon}}}}\right)^{\frac{1}{k^2}} (\sqrt{\varepsilon})^{\frac{k^2-1}{k^2}} + O(\varepsilon).$$





PWL SN-canard

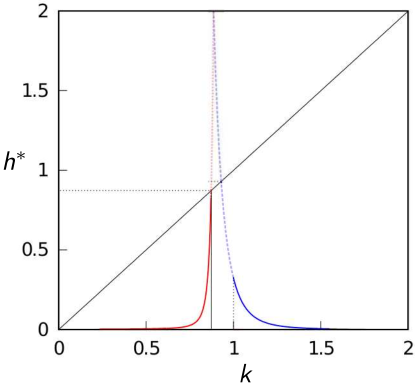


Figure: Amplitude of the saddle-node canard orbit versus parameter k for $\varepsilon = 1e - 6$. The straight line corresponds with the ordinate of the tangent point $\mathbf{p}_L$, and coincides with the maximum size of a 3-zones saddle-node canard orbit. Therefore, the discolored part of the curve does not correspond with saddle-node canard orbits.

PWL SN-canard

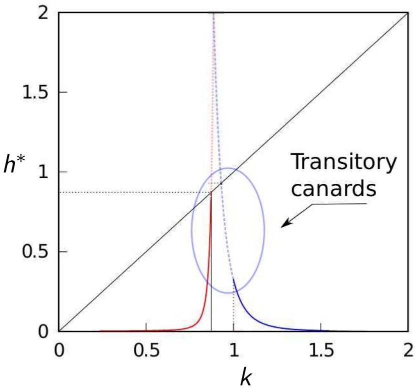


Figure: Amplitude of the saddle-node canard orbit versus parameter k for $\varepsilon = 1e - 6$. The straight line corresponds with the ordinate of the tangent point $\mathbf{p}_L$, and coincides with the maximum size of a 3-zones saddle-node canard orbit. Therefore, the discolored part of the curve does not correspond with saddle-node canard orbits.

Revisiting slow fast dynamics with piecewise linear differential systems.

A.E. Teruel

Sesión I: Piecewise-smooth systems, averaging and hybrid systems.
03/10/2018

CARMONA, FERNÁNDEZ, DESROCHES, A.E.T. *Saddle-node canard orbits in planar PWL system* Work in progress.
CARMONA, FERNÁNDEZ, DESROCHES, A.E.T. *Transitory canards and maximal canard orbits in planar PWL system* Work in progress.
DESROCHES, GUILLAMON, PONCE, PROHENS, RODRIGUES, A.E.T. *Folded nodes and mixed-mode oscillations in piecewise-linear slow-fast systems* SIAM Rev. 2016
FERNÁNDEZ, DESROCHES, KRUPA, A.E.T. *Canard solutions in planar piecewise linear systems with three zones* Dyn. Sys. 2016.
PROHENS, A.E.T. *Canard trajectories in 3D piecewise linear systems* DCDS, 2013