Some contributions to the analysis of piecewise linear systems
PhD thesis

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Advisors:
Enrique Ponce and Javier Ros

Ddays Murcia, October 3-5, 2018.

Outline

1. **Memristor oscillators**
   - Motivation

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   - Main contributions

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     - Application to discontinuous 3D PWL models

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     - Cubic Memristor Oscillator

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   - Publications
2. **Big Bang Bifurcation**

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   - Main contributions

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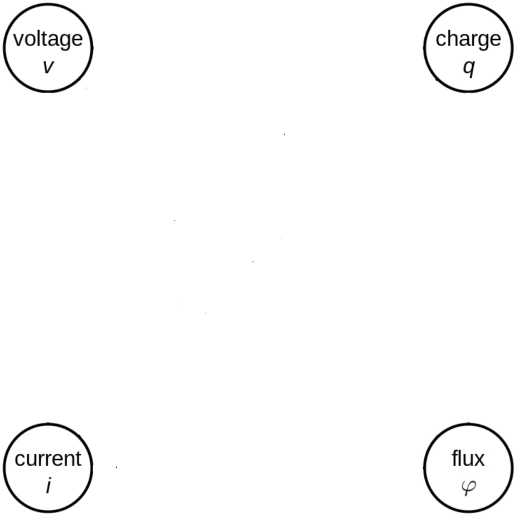
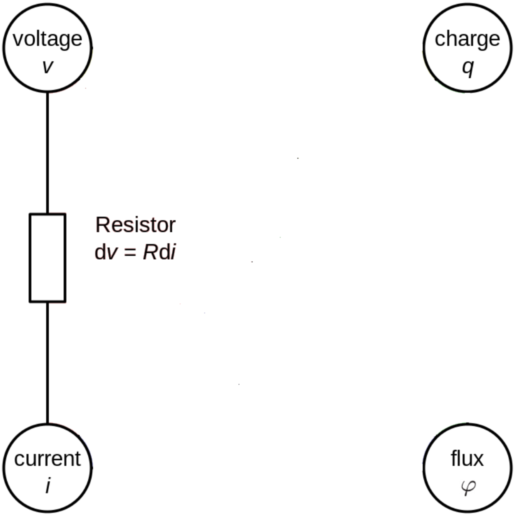
Fundamental elements



Fundamental elements



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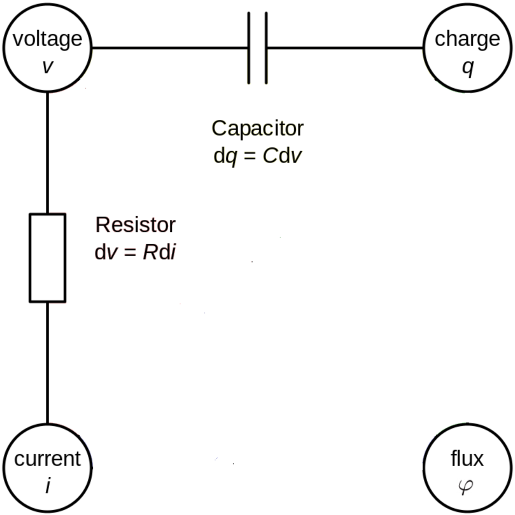

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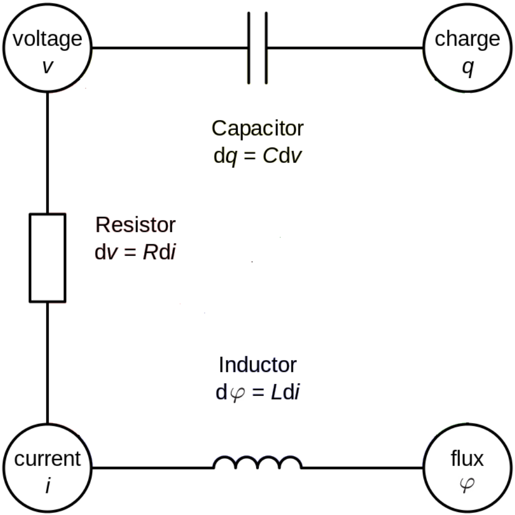
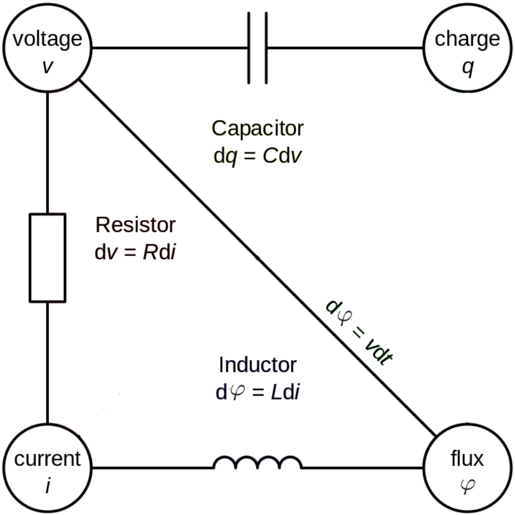
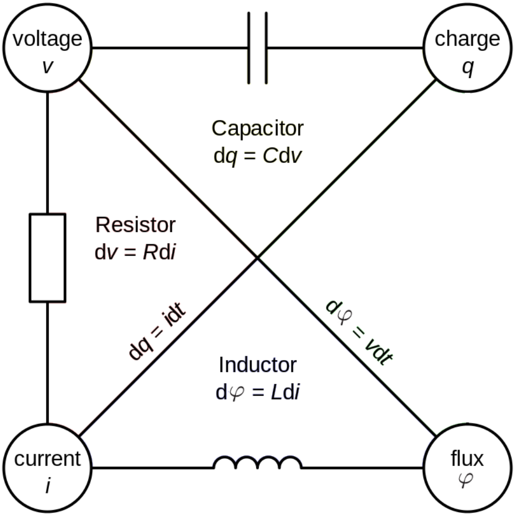


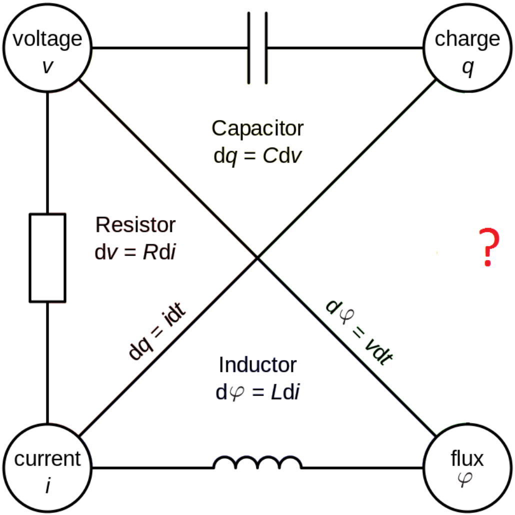
Fundamental elements



Fundamental elements

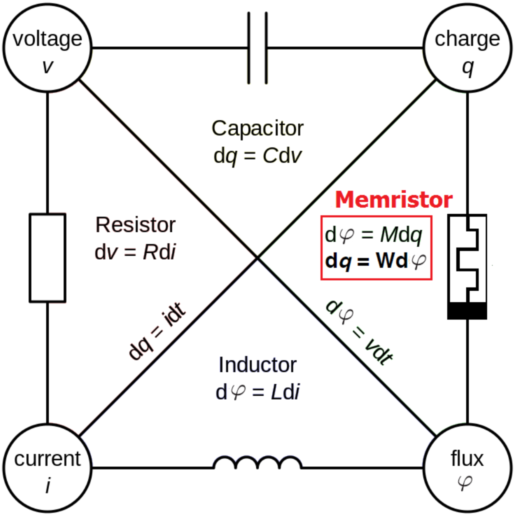


[Chua, 1971]



voltage
v
charge
q
Capacitor
dq = Cdv
Resistor
dv = Rdi
?
dq = idt
dφ = vdt
Inductor
dφ = Ldi
current
i
flux
φ

Memory resistor [Chua, 1971]



Memristor constitutive relation

Definition ([Chua, 1971])

A memristor is a passive two-terminal electronic device, characterized by a nonlinear constitutive relation between the flux ϕ and the charge q.

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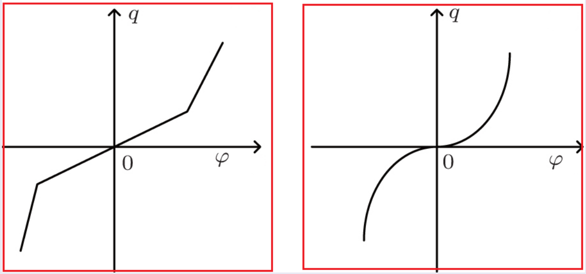

The first memristor

The **missing memristor found**

DB Strukov, GS Snider, DR Stewart, RS Williams - nature, 2008 - nature.com

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nature Vol 453 | 1 May 2008 | doi:10.1038/nature06932

LETTERS

The missing memristor found

Dmitri B. Strukov[1], Gregory S. Snider[1], Duncan R. Stewart[1] & R. Stanley Williams[1]

Memristor oscillators

M Itoh, LO Chua - International Journal of Bifurcation and Chaos, 2008 - World Scientific

MEMRISTOR OSCILLATORS

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Fukuoka 811-0295, Japan

LEON O. CHUA
Department of Electrical Engineering and Computer Sciences,
University of California, Berkeley,
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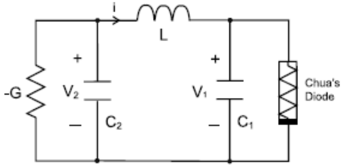
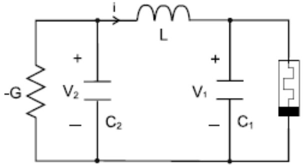
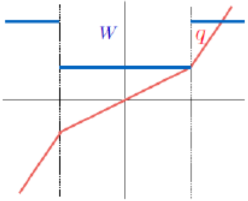
3D memristor



3D memristor





$$\frac{dx}{d\tau} = \alpha\left(y - W(z)x\right),$$

$$\frac{dy}{d\tau} = -\xi x + \beta y,$$

$$\frac{dz}{d\tau} = x,$$

$$\alpha = \frac{1}{C}, \quad \xi = \frac{1}{L}, \quad \beta = \frac{R}{L},$$

$$W(z) = \begin{cases} b, & \text{if} \quad |z| > 1, \\ a, & \text{if} \quad |z| \leq 1, \end{cases}$$

$$q(z) = \begin{cases} b(z-1) + a, & \text{if} \quad z > 1, \\ az, & \text{if} \quad |z| \leq 1, \\ b(z+1) - a, & \text{if} \quad z < -1, \end{cases}$$

Two periodic attractors

$\alpha = 1, \beta = 0.1, \gamma = 1, a = 0.02, b = 2$

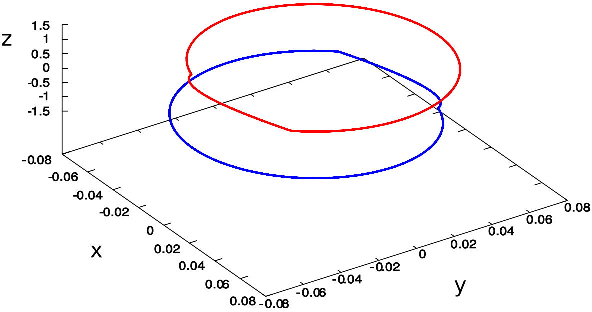
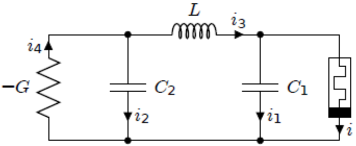


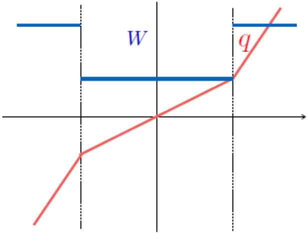
Figure: Two periodic attractors, [Itoh and Chua, 2008]

4D PWL memristor



$$
\begin{aligned}
\frac{dx}{d\tau} &= \alpha y - \alpha W(w)x, \\
\frac{dy}{d\tau} &= z - x, \\
\frac{dz}{d\tau} &= -\beta y + \gamma z, \\
\frac{dw}{d\tau} &= x,
\end{aligned}
$$

$$
\alpha = \frac{1}{C_1}, \quad \beta = \frac{1}{C_2}, \quad \gamma = \frac{G}{C_2}, \quad L = 1
$$



Chaotic attractor

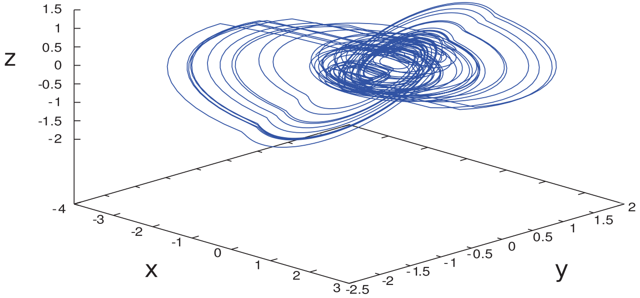


Figure: Chaotic attractor, [Itoh and Chua, 2008]

[Messias et al., 2010] and [Scarabello and Messias, 2014]

International Journal of Bifurcation and Chaos, Vol. 20, No. 2 (2010) 437–450

HOPF BIFURCATION FROM LINES OF EQUILIBRIA WITHOUT PARAMETERS IN MEMRISTOR OSCILLATORS

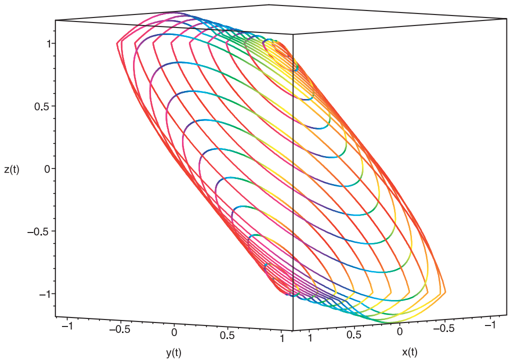
MARCELO MESSIAS*, CRISTIANE NESPOLI† and VANESSA A. BOTTA‡

International Journal of Bifurcation and Chaos, Vol. 24, No. 1 (2014) 1430001 (18 pages)

Bifurcations Leading to Nonlinear Oscillations in a 3D Piecewise Linear Memristor Oscillator

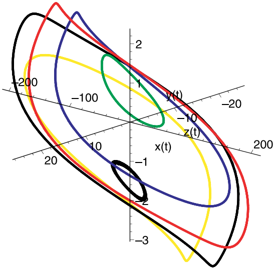
Marluce da Cruz Scarabello* and Marcelo Messias†

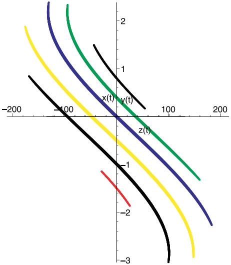
3D PWL memristor oscillator



Cubic memristor

$$q(z) = z^3 + az^2 + bz + c, \quad a^2 - 3b \leq 0$$





Extreme multistability

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Extreme multistability in a memristive circuit
BC Bao, Q Xu, H Bao, M Chen - Electronics Letters, 2016 - IET
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Hidden **extreme multistability** in memristive hyperchaotic system
BC Bao, H Bao, N Wang, M Chen, Q Xu - Chaos, Solitons & Fractals, 2017 - Elsevier
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[HTML] Two-**memristor**-based Chua's hyperchaotic circuit with plane equilibrium and its **extreme multistability**
B Bao, T Jiang, G Wang, P Jin, H Bao, M Chen - Nonlinear Dynamics, 2017 - Springer
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Multiple attractors in a non-ideal active voltage-controlled **memristor** based Chua's circuit
Q Xu, Y Lin, B Bao, M Chen - Chaos, Solitons & Fractals, 2016 - Elsevier
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Outline

The tree-dimensional model

We consider a family of 3D systems, which is general enough to capture all the mathematical models of memristor oscillators.

$$\begin{aligned} \dot{x} &= a_{11}W(z)x + a_{12}y, \\ \dot{y} &= a_{21}x + a_{22}y, \\ \dot{z} &= x, \end{aligned} \qquad (1)$$

where the constants $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$, the function W is defined by

$$W(z) = \frac{dq(z)}{dz}, \qquad (2)$$

and q is a continuous function.

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where the constants $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$, the function W is defined by

$$W(z) = \frac{dq(z)}{dz}, \tag{2}$$

and q is a continuous function.
The equilibrium points of system (1) are given by

$$E = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0 \text{ and } z \in \mathbb{R}\}. \tag{3}$$

Ddays

Theorem

Consider system (1) *where the function W is defined as in* (2)*. For any* $h \in \mathbb{R}$*, the set*

$$S_h = \{(x,y,z) \in \mathbb{R}^3 : -a_{22}x + a_{12}y - a_{12}a_{21}z + a_{11}a_{22}q(z) = h\} \quad (4)$$

is an invariant manifold for the system. Therefore, the system has an infinite family of invariant manifolds foliating the whole $\mathbb{R}^3$*, and so the dynamics is essentially two-dimensional.*

Theorem

Consider system (1) *with function* W *defined as in* (2)*. If* $a_{12} \neq 0$*, then on each invariant set* S_h *given in* (4)*, the dynamics is topologically equivalent to the Liénard system*

$$\dot{X} = Y - F(X), \quad \dot{Y} = -g(X) + h, \tag{5}$$

where F *and* g *are given by*

$$F(X) = -a_{11}q(X) - a_{22}X, \quad g(X) = a_{11}a_{22}q(X) - a_{12}a_{21}X. \tag{6}$$

Theorem

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$$F(X) = -a_{11}q(X) - a_{22}X, \quad g(X) = a_{11}a_{22}q(X) - a_{12}a_{21}X. \tag{6}$$

Moreover, $(X(\tau), Y(\tau)) \in \mathbb{R}^2$ *is a solution of the Liénard system* (5) *for a given* $h \in \mathbb{R}$*, if and only if* $E_h(X(\tau), Y(\tau)) \in \mathbb{R}^3$ *is a solution of system* (1) *on* S_h*, where*

$$E_h(X(\tau), Y(\tau)) = \begin{pmatrix} Y(\tau) - F(X(\tau)) \\ \frac{1}{a_{12}}\left[(a_{22}^2 + a_{12}a_{21})Y(\tau) - a_{22}Y(\tau) + h\right] \\ X(\tau) \end{pmatrix}. \tag{7}$$

3D PWL Memristor Oscillator

Here

$$q(z) = \begin{cases} b(z-1)+a, & \text{if} \quad z > 1, \\ az, & \text{if} \quad |z| \leq 1, \\ b(z+1)-a, & \text{if} \quad z < -1, \end{cases} \tag{8}$$

so that

$$W(z) = \begin{cases} b, & \text{if} \quad |z| > 1, \\ a, & \text{if} \quad |z| \leq 1, \end{cases} \quad \text{with} \quad a \neq b. \tag{9}$$

We define the auxiliary matrices

$$A_E = \begin{pmatrix} b \cdot a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_C = \begin{pmatrix} a \cdot a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{10}$$

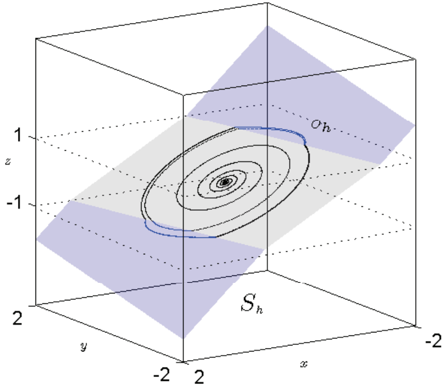
and the traces and determinants of such matrices

$$\begin{aligned} t_E &= b\,a_{11} + a_{22}, & t_C &= a\,a_{11} + a_{22}, \\ d_E &= b\,a_{11}a_{22} - a_{12}a_{21}, & d_C &= a\,a_{11}a_{22} - a_{12}a_{21}. \end{aligned} \tag{11}$$

3D PWL Memristor Oscillator

From Theorem 2, we obtain the piecewise linear invariant manifolds S_h defined by

$$S_h = \begin{cases} a_{12}y - a_{22}x + d_E z = h - a_{22}(t_C - t_E), & \text{if} \quad z > 1, \\ a_{12}y - a_{22}x + d_C z = h, & \text{if} \quad |z| \leq 1, \\ a_{12}y - a_{22}x + d_E z = h - a_{22}(t_E - t_C), & \text{if} \quad z < -1, \end{cases} \tag{12}$$



Theorem

Consider the function W defined as in (9). If $a_{12} \neq 0$, then on each invariant set S_h of system (1), the dynamics is topologically equivalent to the continuous Liénard system

$$\dot{X} = F(X) - Y, \quad \dot{Y} = g(X) - h, \tag{13}$$

where F and g are given by

$$F(X) = \begin{cases} t_E(X-1)+t_C, & \text{if} \quad X>1, \\ t_C X, & \text{if} \quad |X| \leq 1, \\ t_E(X+1)-t_C, & \text{if} \quad X<-1, \end{cases} \tag{14}$$

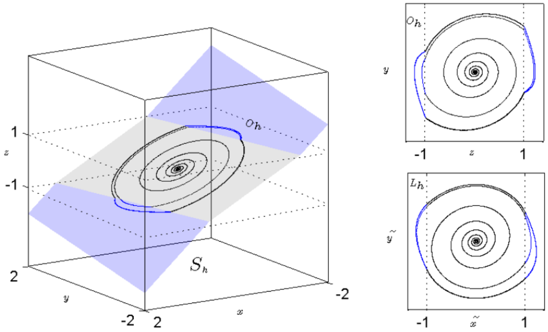
$$g(X) = \begin{cases} d_E(X-1)+d_C, & \text{if} \quad X>1, \\ d_C X, & \text{if} \quad |X| \leq 1, \\ d_E(X+1)-d_C, & \text{if} \quad X<-1, \end{cases} \tag{15}$$

and t_E, t_C, d_E, d_C are the traces and determinants (11) of the matrices defined in (10).

Moreover, $(X(\tau), Y(\tau)) \in \mathbb{R}^2$ is a solution of the continuous reduced system (13) for a given $h \in \mathbb{R}$, if and only if $E_h(X(\tau), Y(\tau)) \in \mathbb{R}^3$ is a solution of system (1) on S_h, where

$$E_h(X(\tau), Y(\tau)) = \begin{pmatrix} F(X(\tau)) - Y(\tau) \\ \frac{1}{a_{12}}\left[(a_{22}^2 + a_{12}a_{21})X(\tau) - a_{22}Y(\tau) + h\right] \\ X(\tau) \end{pmatrix}, \quad (16)$$

and the function F is defined as in (14).

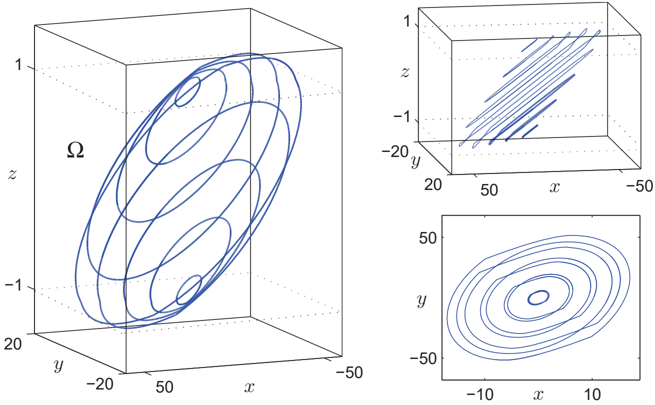


Theorem

Consider system (1)-(9) *with* $a \neq b$, *and parameters such that*

$$t_E < 0, \quad t_C > 0, \quad d_E, d_C > 0, \tag{17}$$

where t_E, t_C, d_E *and* d_C *are the traces and determinants of matrices defined in* (10). *Then for any* $|h| < d_C$ *the system has an infinite number of stable periodic orbits, each one of them contained in the set* S_h *defined in* (12),
Moreover, such periodic orbits generate a tubular surface Ω *which is homeomorphic to the cylinder* $\mathbf{S^1} \times (-1, 1)$, *and symmetric with respect to the origin.*



1
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1
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x
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50
y
−50
−10
x
10

Cubic Memristor Oscillator

Theorem

Consider system (1) *with the functions q and W defined by*

$$\begin{aligned} q(z) &= cz^3 + az^2 + bz, \\ W(z) &= q'(z) = 3cz^2 + 2az + b. \end{aligned} \tag{18}$$

Cubic Memristor Oscillator

Theorem

Consider system (1) *with the functions* q *and* W *defined by*

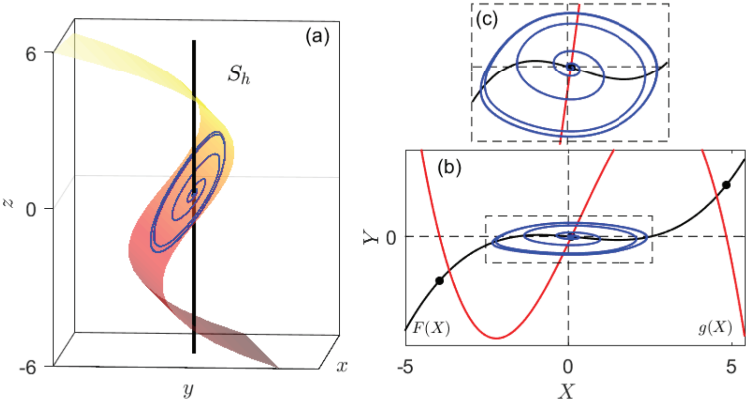
$$\begin{aligned} q(z) &= cz^3 + az^2 + bz, \\ W(z) &= q'(z) = 3cz^2 + 2az + b. \end{aligned} \tag{18}$$

If $a_{12} \neq 0$, *then on each invariant set* S_h *given by*

$$\begin{aligned} S_h = \{(x,y,z) \in \mathbb{R}^3 : -a_{22}x + a_{12}y + a_{11}a_{22}cz^3 + \\ + aa_{11}a_{22}z^2 + (ba_{11}a_{22} - a_{12}a_{21})z = h\} \end{aligned} \tag{19}$$

the dynamics is topologically equivalent to the Liénard system

$$\begin{aligned} \dot{x} &= y + ca_{11}x^3 + aa_{11}x^2 + (ba_{11} + a_{22})x, \\ \dot{y} &= -a_{11}a_{22}cx^3 - a_{11}a_{22}ax^2 + (a_{12}a_{21} - a_{11}a_{22}b)x + h. \end{aligned} \tag{20}$$


(a)
S_h
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0
-6
z
y
x
(c)
(b)
Y 0
$F(X)$
$g(X)$
-5
0
5
X

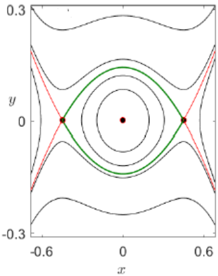
Moreover, if $a_{11}a_{22} < 0$ then the system can be written into the form

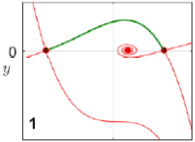
$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 + \mu_2 x + cx^3 + y(\mu_3 + 3ca_{11}x^2), \end{aligned} \tag{21}$$

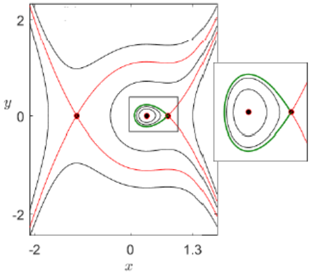
where the new parameters μ_1, μ_2 and μ_3 are

$$\mu_1 = \frac{27ch + a_{11}a_{22}a(9cb - 2a^2) - 9caa_{12}a_{21}}{27c^2\left(-a_{11}a_{22}\right)^{5/2}},$$

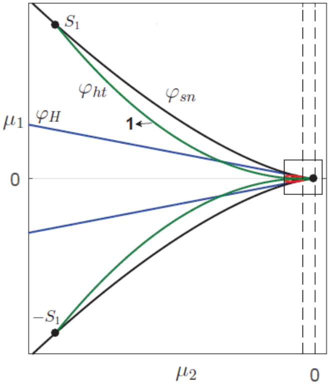
$$\mu_2 = \frac{a_{11}a_{22}(a^2 - 3cb) + 3ca_{12}a_{21}}{3c\left(a_{11}a_{22}\right)^2}, \quad \mu_3 = \frac{a_{11}(a^2 - 3cb) - 3ca_{22}}{3ca_{11}a_{22}}.$$







$$\dot{x} = y,$$
$$\dot{y} = \mu_1 + \mu_2 x + x^3 + y(\mu_3 - 3x^2),$$



$$\varphi_{sn} = \{(\mu_1, \mu_2, \mu_3) : 27\mu_1^2 + 4\mu_2^3 = 0 \ \ and \ \mu_3 \in \mathbb{R}\},$$

$$\varphi_H = \{(\mu_1, \mu_2, \mu_3) : \mu_1 = \mp \left(\frac{\mu_3}{3}\right)^{3/2} \mp \left(\frac{\mu_3}{3}\right)^{1/2} \mu_2, \ \mu_2 < -\mu_3\},$$

Melnikov theory for $\mu_3 > 0$

$$\varphi_h = \{(\mu_2, \mu_1) \in \mathbb{R}^2 : \mu_1 = \mu_3 \nu_2(\theta), \quad \mu_2 = \pm \mu_3^{3/2} \nu_1(\theta), \quad 0 < \theta < \infty\},$$

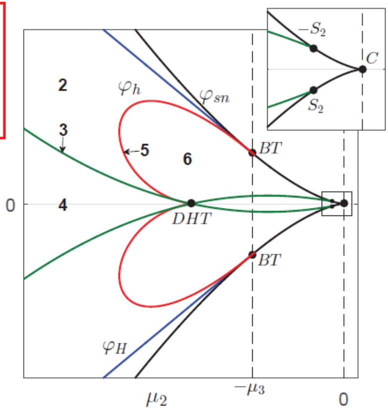
$$\nu_2(\theta) = \frac{-10(\cosh 2\theta + 5)(9 \sinh \theta + \sinh 3\theta - 12\theta \cosh \theta)}{3(370 \sinh \theta + 115 \sinh 3\theta + \sinh 5\theta - 60\theta(11 \cosh \theta + \cosh 3\theta))},$$

$$\nu_1(\theta) = -(\nu_2(\theta) s + s^3), \quad s^2 = \frac{-\cosh^2 \theta}{2 + \cosh^2 \theta} \nu_2(\theta).$$

$$\varphi_{ht} = \{(\mu_2, \mu_1) \in \mathbb{R}^2 : \mu_1 = \pm \frac{\sqrt{2}}{15} \mu_2 (3\mu_2 + 5\mu_3), \quad \mu_2 < -\frac{5}{3}\mu_3\}.$$

$$DHT \equiv (0, -5\mu_3/3).$$

$$BT_\pm \equiv \left(-\mu_3, \pm \frac{2}{3} \sqrt{\frac{\mu_3^3}{3}}\right).$$



Application to a cubic memristor

Theorem

Consider the cubic memristor with $(a^2 - 3b + 3\beta)/(3\beta) > 0$ *sufficiently small, and suppose that* $0 < 3b - a^2 < 3\xi/\beta$.
Then there exists $K < 0$ *with* $K < (-5/3)(a^2 - 3b + 3\beta) < 0$*, such that the system has an infinite number of stable periodic orbits; in particular, for any initial condition* $(x_0, y_0, z_0) \in \mathbb{R}^3$ *with* $x_0 \neq 0$ *or* $y_0 \neq 0$ *and*

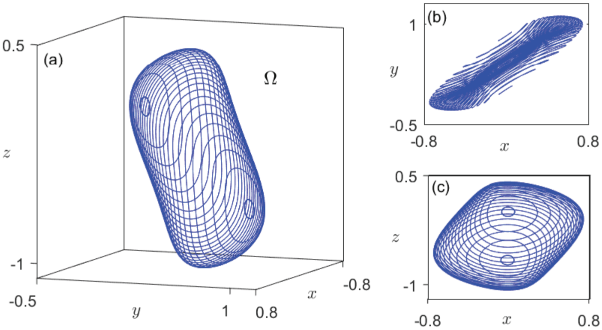
$$\min\{A, B\} < -\beta x_0 - \xi y_0 + \xi z_0 - \beta q(z_0) < \max\{A, B\},$$

where

$$A = \frac{1}{27}\left(a - \sqrt{a^2 - 3b + 3\beta}\right)\left(6b\beta - 9\xi + 3\beta^2 - a^2\beta + a\beta\sqrt{a^2 - 3b + 3\beta}\right)$$

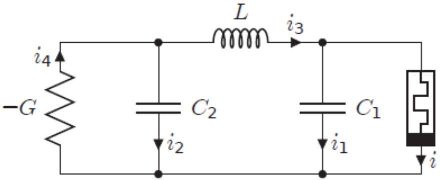
$$B = -\frac{1}{27}\left(a + \sqrt{a^2 - 3b + 3\beta}\right)\left(9\xi - 6b\beta - 3\beta^2 + a^2\beta + a\beta\sqrt{a^2 - 3b + 3\beta}\right),$$

the steady state solution is periodic. Moreover, the periodic orbits generate a topological sphere Ω *foliated by such periodic orbits.*

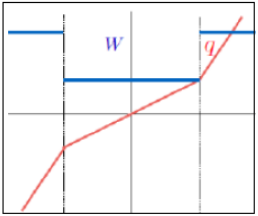


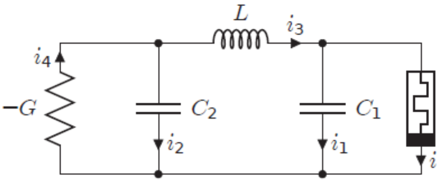
0.5
(a)
Ω
z
-1
-0.5
y
1
0.8
x
-0.8
(b)
1
y
-0.5
-0.8
x
0.8
(c)
0.5
z
-1
-0.8
x
0.8

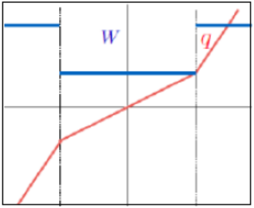
Application to a 4D memristor oscillator



$$\frac{dx}{d\tau} = \alpha y - \alpha W(w)x,$$
$$\frac{dy}{d\tau} = z - x,$$
$$\frac{dz}{d\tau} = -\beta y + \gamma z,$$
$$\frac{dw}{d\tau} = x,$$







$$\frac{dx}{d\tau} = \alpha y - \alpha W(w)x,$$
$$\frac{dy}{d\tau} = z - x,$$
$$\frac{dz}{d\tau} = -\beta y + \gamma z,$$
$$\frac{dw}{d\tau} = x,$$

$$\dot{y}_1 = y_4,$$
$$\dot{y}_2 = y_3 - y_4,$$
$$\dot{y}_3 = -\beta y_2 + \gamma y_3,$$
$$\dot{y}_4 = y_2 - W(y_1)y_4,$$

$$\begin{aligned}
\dot{y}_1 &= y_4, \\
\dot{y}_2 &= y_3 - y_4, \\
\dot{y}_3 &= -\beta y_2 + \gamma y_3, \\
\dot{y}_4 &= y_2 - W(y_1) y_4,
\end{aligned}$$

$$H(y_1, y_2, y_3, y_4) = \beta\,(y_4 + q(y_1)) - \gamma\,(y_1 + y_2) + y_3,$$

$$S_h = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : H(y_1, y_2, y_3, y_4) = h\}.$$

$$\begin{aligned}
\dot{x}_1 &= \frac{\gamma}{\beta}(x_1 + x_2) - q(x_1) - \frac{1}{\beta} x_3 + \frac{h}{\beta}, \\
\dot{x}_2 &= x_3 - \frac{\gamma}{\beta}(x_1 + x_2) + q(x_1) + \frac{1}{\beta} x_3 - \frac{h}{\beta}, \\
\dot{x}_3 &= -\beta x_2 + \gamma x_3.
\end{aligned}$$

$$\dot{x}_1 = \frac{\gamma}{\beta}(x_1 + x_2) - q(x_1) - \frac{1}{\beta}x_3 + \frac{h}{\beta},$$
$$\dot{x}_2 = x_3 - \frac{\gamma}{\beta}(x_1 + x_2) + q(x_1) + \frac{1}{\beta}x_3 - \frac{h}{\beta},$$
$$\dot{x}_3 = -\beta x_2 + \gamma x_3.$$

$$A_C = \frac{1}{\beta}\begin{pmatrix} \gamma - a\beta & \gamma & -1 \\ a\beta - \gamma & -\gamma & 1+\beta \\ 0 & -\beta^2 & \gamma\beta \end{pmatrix},$$
$$A_E = \frac{1}{\beta}\begin{pmatrix} \gamma - b\beta & \gamma & -1 \\ b\beta - \gamma & -\gamma & 1+\beta \\ 0 & -\beta^2 & \gamma\beta \end{pmatrix},$$

$$\dot{\mathbf{x}} = \begin{pmatrix} t_E & -1 & 0 \\ m_E & 0 & -1 \\ d_E & 0 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t_C - t_E \\ m_C - m_E \\ d_C - d_E \end{pmatrix}\operatorname{sat}\left(\mathbf{e}_1^T\mathbf{x}\right) + \frac{h}{\beta}\begin{pmatrix} 1 \\ \gamma \\ \beta \end{pmatrix}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} t_E & -1 & 0 \\ m_E & 0 & -1 \\ d_E & 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t_C - t_E \\ m_C - m_E \\ d_C - d_E \end{pmatrix} \operatorname{sat}\left(\mathbf{e}_1^T \mathbf{x}\right) + \frac{h}{\beta} \begin{pmatrix} 1 \\ \gamma \\ \beta \end{pmatrix}$$

$h \in \mathbb{R}$ *and* $\beta \neq 0, \qquad (x_1(\tau), x_2(\tau), x_3(\tau)) \in \mathbb{R}^3$

$$\mathbf{y}(\tau) = \begin{pmatrix} x_1(\tau) \\ (\gamma^2 - \beta - 1)x_1(\tau) - \gamma x_2(\tau) + x_3(\tau) \\ (\gamma^3 - 2\beta\gamma)x_1(\tau) + (\beta - \gamma^2)x_2(\tau) + \gamma x_3(\tau) \\ \gamma x_1(\tau) - x_2(\tau) - q(x_1(\tau)) + h/\beta \end{pmatrix}$$

Theorem

Take $m_C > 0$, $\varepsilon = m_C t_C - d_C$, and assume the non-degeneracy condition

$$\rho = d_C m_C - d_C m_E + d_E m_C - m_C^2 t_E \neq 0.$$

Then, for $\varepsilon = 0$ the 4D system undergoes a focus-center-limit cycle bifurcation simultaneously on all the levels S_h with $|h| < |d_C| \neq 0$.

Thus, from the lineal center configurations in the central zone, which exist for $\varepsilon = 0$, an infinite number of stable periodic orbits simultaneously appears for $\varepsilon\rho > 0$ and ε sufficiently small.

Theorem

Take $m_C > 0$, $\varepsilon = m_C t_C - d_C$, and assume the non-degeneracy condition

$$\rho = d_C m_C - d_C m_E + d_E m_C - m_C^2 t_E \neq 0.$$

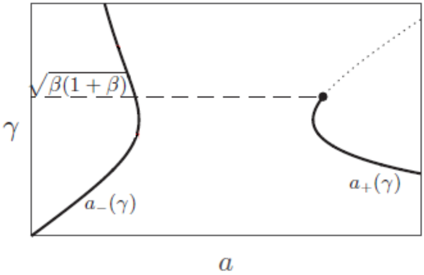
Then, for $\varepsilon = 0$ the 4D system undergoes a focus-center-limit cycle bifurcation simultaneously on all the levels S_h with $|h| < |d_C| \neq 0$.

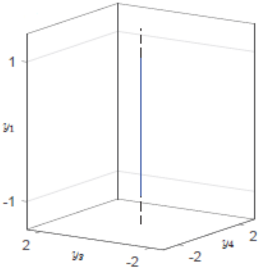
In particular, if $\rho > 0$ and $d_C < 0$, then the periodic orbits bifurcates for $\varepsilon > 0$ and are orbitally asymptotically stable. Otherwise, the bifurcating periodic orbits are unstable.

MFCC bifurcation

$$a_{\pm}(\gamma) = \frac{\gamma^2+1}{2\gamma} \pm \sqrt{\left(\frac{\gamma^2+1}{2\gamma}\right)^2 - \beta},$$

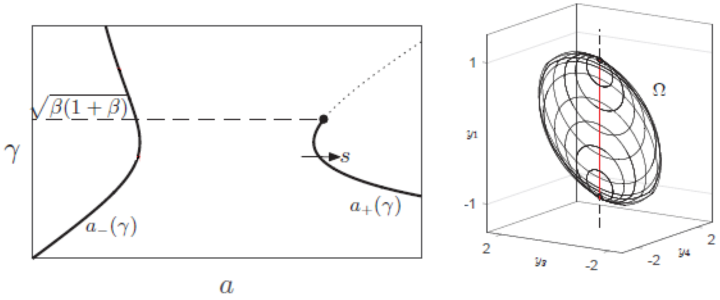
If $0 < \beta < 1$, $\gamma < \sqrt{\beta(1+\beta)}$ and $0 < b < a$ then for $a = a_+(\gamma)$ the system undergoes a MFCC bifurcation, so that when $a \leq a_+(\gamma)$ all the equilibria in central segment are stable, becoming unstable for $a > a_+(\gamma)$.

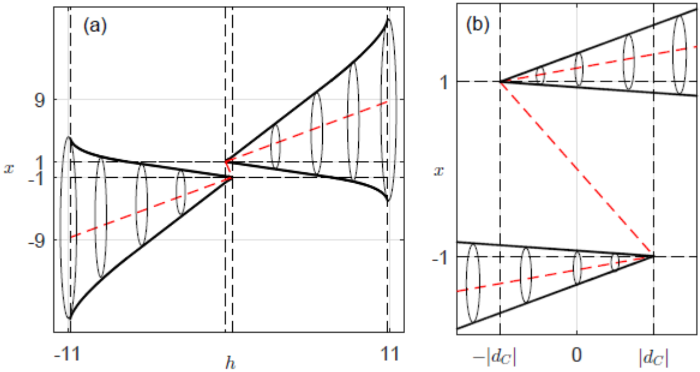




MFCC bifurcation

For $a - a_+(\gamma) > 0$ and sufficiently small, there appears a bounded hypersurface $\Omega \subset \mathbb{R}^4$ foliated by stable periodic orbits.




(a)
9
1
-1
-9
x
-11
h
11
(b)
1
-1
x
$-|d_C|$
0
$|d_C|$

Outline

1. **Memristor oscillators**
   - Motivation
   - Main contributions
     - Application to discontinuous 3D PWL models
     - Cubic Memristor Oscillator
     - Application to a 4D memristor oscillator
   - Publications
2. **Big Bang Bifurcation**
   - Motivation
   - Main contributions
   - Publications
3. **References**

Publications

- On Discontinuous Piecewise Linear Models for Memristor Oscillators, *International Journal of Bifurcation and Chaos*. DOI: 10.1142/s0218127417300221.
- Unravelling the dynamical richness of 3D canonical memristor oscillators, *Microelectronic Engineering*. DOI:10.1016/j.mee.2017.08.004.
- Bifurcation set of a Bogdanov-Takens system with symmetry. Application to 3D cubic Memristor oscillators, *Submitted* .
- A multiple focus-center-cycle bifurcation in 4D discontinuous piecewise linear Memristor oscillators, *Nonlinear Dynamics*. DOI: 10.1007/s11071-018-4541-2.

Outline

Differ Equ Dyn Syst (Jan&Apr 2013) 21(1&2):35–43
DOI 10.1007/s12591-012-0121-y

ORIGINAL RESEARCH

Occurrence of Big Bang Bifurcations in Discretized Sliding-mode Control Systems

Enric Fossas · Albert Granados

The authors consider a system with a relay based control and a linear switching manifold defined by

$$\dot{\mathbf{x}} = \begin{cases} A\mathbf{x} + \mathbf{b}, & \text{if} \quad \sigma(\mathbf{x}) \geq 0, \\ A\mathbf{x} - \mathbf{b}, & \text{if} \quad \sigma(\mathbf{x}) < 0, \end{cases}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ bk \end{pmatrix},$$

and

$$\sigma(\mathbf{x}) = \mathbf{e}_1^T \mathbf{x} + c\mathbf{e}_2^T \mathbf{x} - y_c.$$

[Fossas and Granados, 2013]

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} A\mathbf{x} + \mathbf{b}_L, & \text{if} \quad \mathbf{e}_1^T \mathbf{x} \leq 0, \\ A\mathbf{x} + \mathbf{b}_R, & \text{if} \quad \mathbf{e}_1^T \mathbf{x} > 0, \end{cases} \tag{22}$$

where matrix A and vectors $\mathbf{b}_{\{L,R\}}$ are

$$A = \begin{pmatrix} -ca_0 & a_0 c^2 - a_1 c + 1 \\ -a_0 & ca_0 - a_1 \end{pmatrix}, \ \mathbf{b}_R = \begin{pmatrix} bkc - ca_0 y_c \\ bk - a_0 y_c \end{pmatrix}, \ \mathbf{b}_L = \begin{pmatrix} -bkc - ca_0 y_c \\ -bk - a_0 y_c \end{pmatrix},$$

and $\mathbf{e}_1$ is the first canonical vector.

Stroboscopic map

For $|A| = a_0 \neq 0$, a fixed $t > 0$ and taking into account the solutions of each vector field, the stroboscopic map is defined by

$$P(\mathbf{x};t) = \begin{cases} e^{At}\mathbf{x} + (e^{At} - I)A^{-1}\mathbf{b}_L, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} \leq 0, \\ e^{At}\mathbf{x} + (e^{At} - I)A^{-1}\mathbf{b}_R, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} > 0, \end{cases} \tag{23}$$

Stroboscopic map

For $|A| = a_0 \neq 0$, a fixed $t > 0$ and taking into account the solutions of each vector field, the stroboscopic map is defined by

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If $bck \neq 0$ the map is discontinuous, and always has two fixed points given by

$$\mathbf{x}^*_{\{L,R\}} = -A^{-1}\mathbf{b}_{\{L,R\}}$$

Big Bang Bifurcation

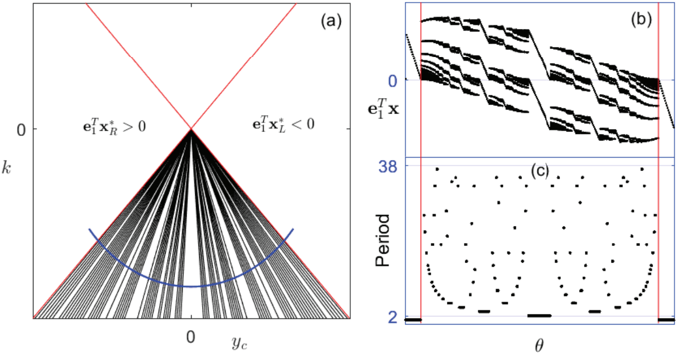


Figure: $t = 0.1$, $a_0 = -2$, $a_1 = -5$, $b = -1$ and $c = 1.5$.

Big Bang Bifurcation

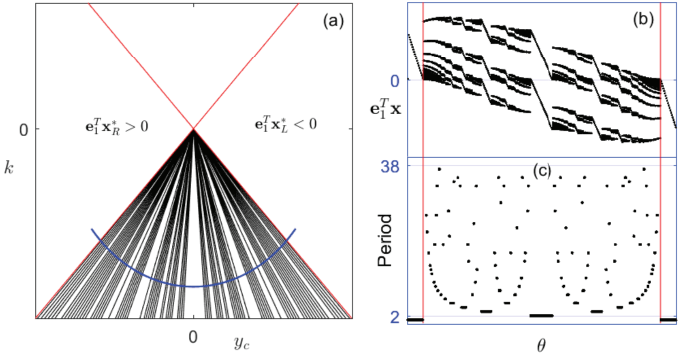


Figure: $t = 0.1$, $a_0 = -2$, $a_1 = -5$, $b = -1$ and $c = 1.5$.

They conjectured that when the eigenvalues of the matrix e^{At} are real and lower than 1, both fixed points are virtual and the sliding set is attractive, then the stroboscopic map has a BB bifurcation point at $(y_c, k) = (0, 0)$.

Normalized canonical form [Freire et al., 2014]

Consider the planar piecewise linear system

$$\dot{\mathbf{x}} = \begin{cases} A_{-}\mathbf{x} + \mathbf{b}_{-}, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} \leq 0, \\ A_{+}\mathbf{x} + \mathbf{b}_{+}, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} > 0, \end{cases} \tag{24}$$

where $A_{\pm} = (a_{ij}^{\pm})$ are constant matrices of order 2. If we consider the modal parameter $m_{\{L,R\}} \in \{0, 1, i\}$ defined in each zone by

$$m_{\{L,R\}} = \begin{cases} i, & \text{if} \quad t_{\mp}^2 - 4d_{\mp} < 0, \\ 0, & \text{if} \quad t_{\mp}^2 - 4d_{\mp} = 0, \\ 1 & \text{if} \quad t_{\mp}^2 - 4d_{\mp} > 0. \end{cases}$$

where i is the unit imaginary.

Normalized canonical form [Freire et al., 2014]

Then the system can be written into the normalized canonical form defined by

$$\dot{\mathbf{x}} = \begin{cases} A_L\mathbf{x} - \mathbf{b}_L, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} \leq 0, \\ A_R\mathbf{x} - \mathbf{b}_R, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} > 0, \end{cases} \tag{25}$$

where

$$A_j = \begin{pmatrix} 2\gamma_j & -1 \\ \gamma_j^2 - m_j^2 & 0 \end{pmatrix}, \quad \mathbf{b}_R = \begin{pmatrix} -b \\ a_R \end{pmatrix}, \quad \mathbf{b}_L = \begin{pmatrix} 0 \\ a_L \end{pmatrix}, \quad j = \{L, R\}. \tag{26}$$

Ddays

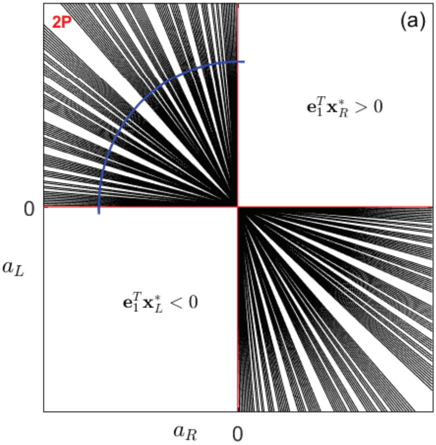
Outline

The stroboscopic map when $A_L = A_R$

Given $m \in \{0, 1, i\}$ and a fixed value $t > 0$, the stroboscopic map P is defined by

$$P(\mathbf{x}; t) = \begin{cases} P_L(\mathbf{x}; t) = e^{At}\mathbf{x} - (\Phi - I)A^{-1}\mathbf{b}_L, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} \leq 0, \\ P_R(\mathbf{x}; t) = e^{At}\mathbf{x} - (\Phi - I)A^{-1}\mathbf{b}_R, & \text{if} \quad \mathbf{e}_1^T\mathbf{x} > 0, \end{cases}$$

where the matrix A is defined as in (26).



2P
(a)
$\mathbf{e}_1^T \mathbf{x}_R^* > 0$
$\mathbf{e}_1^T \mathbf{x}_L^* < 0$
0
a_L
a_R
0

To alleviate notation, we define the auxiliary functions

$$C_k = \begin{cases} \cosh(kt), & \text{if} \quad m = 1, \\ \cos(kt), & \text{if} \quad m = i, \\ 1, & \text{if} \quad m = 0, \end{cases} \quad , \quad S_k = \begin{cases} \sinh(kt), & \text{if} \quad m = 1, \\ \sin(kt), & \text{if} \quad m = i, \\ kt, & \text{if} \quad m = 0. \end{cases}$$

and for $k \geq 1$, and $m \in \{0, 1, i\}$,

$$\mu_k^{\pm} := C_k \pm \gamma S_k.$$

Theorem

Given $m \in \{0, 1, i\}$, $0 < t < 1$, $\gamma \in \mathbb{R}$ such that $D = \gamma^2 - m^2 > 0$ and $\gamma < 0$, consider the functions

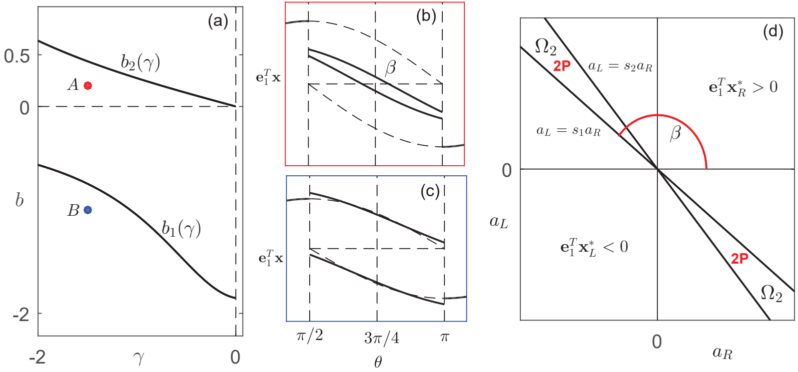
$$h_2^{(1)}(\gamma) = \frac{e^{4t\gamma} - e^{3t\gamma}\mu_1^+ - e^{2t\gamma}\mu_2^- + e^{t\gamma}\mu_1^-}{1 + e^{4t\gamma} - 2C_2 e^{2t\gamma}},$$

$$r_2^{(1)}(\gamma) = \frac{e^{2t\gamma}S_2 - \left(e^{3t\gamma} + e^{t\gamma}\right)S_1}{1 + e^{4t\gamma} - 2C_2 e^{2t\gamma}},$$

and

$$b_1(\gamma) = \frac{h_2^{(1)}(\gamma)}{r_2^{(1)}(\gamma)\, D}, \quad b_2(\gamma) = \frac{2h_2^{(1)}(\gamma) - 1}{\sqrt{2}\, r_2^{(1)}(\gamma)\, D}.$$

The following statements hold for map P.



(a)
0.5
0
-2
b
$b_2(\gamma)$
$b_1(\gamma)$
A
B
-2
γ
0
(b)
(c)
$\mathbf{e}_1^T\mathbf{x}$
β
$\mathbf{e}_1^T\mathbf{x}$
$\pi/2$
$3\pi/4$
π
θ
(d)
Ω_2
2P
$a_L = s_2 a_R$
$\mathbf{e}_1^T\mathbf{x}_R^* > 0$
$a_L = s_1 a_R$
β
0
a_L
$\mathbf{e}_1^T\mathbf{x}_L^* < 0$
2P
Ω_2
0
a_R

(a) For all $b \in \mathbb{R}$ with $b_1(\gamma) < b < b_2(\gamma)$ there exists a unique $\beta \in (3\pi/4, \pi)$ defined by

$$\beta = \arcsin\left(\frac{-br_2^{(1)}(\gamma)\,D}{\sqrt{(1-h_2^{(1)}(\gamma))^2+(h_2^{(1)}(\gamma))^2}}\right) - \pi - \arctan\left(\frac{h_2^{(1)}(\gamma)}{1-h_2^{(1)}(\gamma)}\right),$$

such that for all $(a_R, a_L) \in \Omega_2$ map P has a unique stable 2-periodic orbit, where

$$\Omega_2 = \{(a_R, a_L) \in \mathbb{R}^2 : s_2 a_R < a_L < s_1 a_R, \quad s_1 a_R < a_L < s_2 a_R\},$$

with $s_1 = \tan(\beta)$ and $s_2 = 1/s_1$.

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with $s_1 = \tan(\beta)$ and $s_2 = 1/s_1$.

(b) For all $b \in \mathbb{R}$ with $b \leq b_1(\gamma)$ map P has a unique stable 2-periodic orbit for all $a_R, a_L \in \mathbb{R}$ with $a_R \cdot a_L < 0$.

Big Bang Bifurcation

Conjecture

Given $m \in \{0,1\}$ and $0 < t < 1$. Consider the two-parameter plane (a_R, a_L), and the functions

$$h_3^{(2)}(\gamma,t) = \frac{e^{4t\gamma}\mu_2^+ - e^{3t\gamma}\mu_3^+ - e^{t\gamma}\mu_1^- + 1}{e^{6t\gamma} - 2C_3 e^{3t\gamma} + 1},$$

$$r_3^{(2)}(\gamma,t) = \frac{e^{4t\gamma}S_2 - e^{3t\gamma}S_3 + e^{t\gamma}S_1}{e^{6t\gamma} - 2C_3 e^{3t\gamma} + 1}.$$

Big Bang Bifurcation

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$$r_3^{(2)}(\gamma,t) = \frac{e^{4t\gamma}S_2 - e^{3t\gamma}S_3 + e^{t\gamma}S_1}{e^{6t\gamma} - 2C_3 e^{3t\gamma} + 1}.$$

Then for $\gamma + m < 0$ sufficiently small and

$$b = \min\left\{\frac{1}{\gamma+m}, F(\gamma,t)\right\},$$

where F is defined by

$$F(\gamma,t) = \frac{1}{\gamma^2 - m^2}\frac{2h_3^{(2)}(\gamma,t) - 1}{\sqrt{2}r_3^{(2)}(\gamma,t)},$$

the stroboscopic map P has at $(0,0)$ a big bang bifurcation point of codimension two.

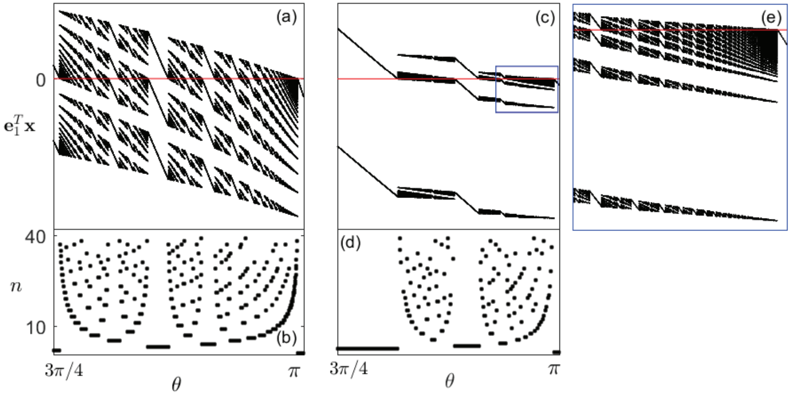

Figure: (a) $m = 0, t = 0.9$, $\gamma = -0.2$ and $b = 1/\gamma$. (c) $m = 1$, $t = 0.9$, $\gamma = -1.05$ and $b = F(\gamma, t) \approx -9.27$.

Outline

1. **Memristor oscillators**
   - Motivation
   - Main contributions
     - Application to discontinuous 3D PWL models
     - Cubic Memristor Oscillator
     - Application to a 4D memristor oscillator
   - Publications
2. **Big Bang Bifurcation**
   - Motivation
   - Main contributions
   - Publications
3. **References**

Publications

- On the Big Bang Bifurcation in the stroboscopic map for discontinuous PWL systems. An application in Discretized Sliding-mode Control Systems. *In preparation*.

¡¡¡ Thanks for your attention !!!

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