Avances en el estudio de propiedades topológicas de flujos analíticos en superficies

José Ginés Espín Buendía

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Structure of the presentation

1 Introduction

The qualitative theory of differential equations: the beginning

Henri Poincaré (1854-1912) proposed a new approach to the study of differential equations: to try to understand the behaviour of the solution of an equation without resolving it.

Vector fields vs. flows

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(If Φ is differentiable with respect t)
$f(z) = \frac{\partial}{\partial t}\Phi(0,z)$ $\nwarrow$

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Two flows Φ_1, Φ_2 on S are called topologically equivalent if there exists a homeomorphism $h : S \to S$ taking orbits onto orbits and preserving the time directions.

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THE GOAL OF OUR THESIS:

To investigate, up to topological equivalence, the effects of analyticity in that asymptotic behaviour.

2 ω-limit sets fon analytic flows on the sphere

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Generalization of Poincaré-Bendixson

Theorem (Vinograd,1952)

$\Omega \subset \mathbb{S}^2$ is the ω-limit set of an orbit of some C^0 (which can be got C^∞) flow on $\mathbb{S}^2$ if, and only if, Ω is the boundary (in $\mathbb{S}^2$) of some simply connected region $O \subset \mathbb{S}^2$ (i. e. an open connected subset $O \subset \mathbb{S}^2$ such that $\mathbb{S}^2 \setminus O$ is also connected).

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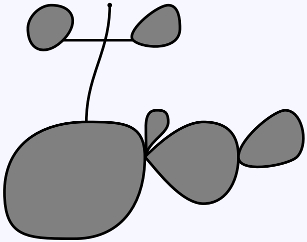

Figure 1: This is an ω-limit. . .

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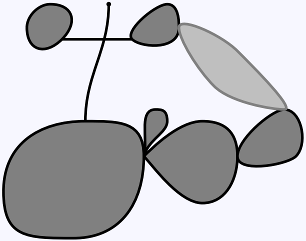

Figure 1: . . . but this is not.

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AIM OF THE SECTION. To come back to Poincaré's hypotheses: to work with analytic flows and study the properties of the ω-limit sets.

Analytic case on the sphere

Theorem (Jiménez and Llibre)

If Φ is an analytic flow on $\mathbb{S}^2$, then any ω-limit set of Φ is either one (singular) point, or the boundary of a *cactus* C (a connected union of finitely many topological disks with connected complementary).

Conversely, if C is a cactus, then there are an analytic flow Φ and a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\mathrm{Bd}\, h(C)$ is an ω-limit set of Φ.

V. Jiménez and J. Llibre, *A topological characterization of the ω-limit sets for analytic flows on the plane, the sphere and the projective plane*, Adv. Math., 2007.

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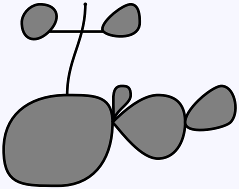

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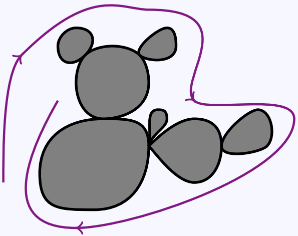

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But . . .

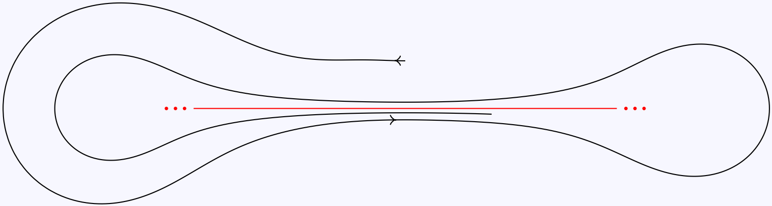

Figure 2: Auxiliary lemma

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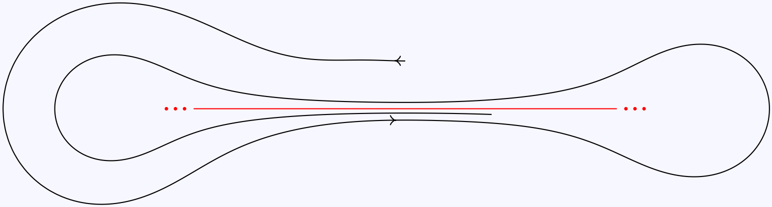

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- For the sphere (the plane and the projective plane), the lemma and the characterization are correct.

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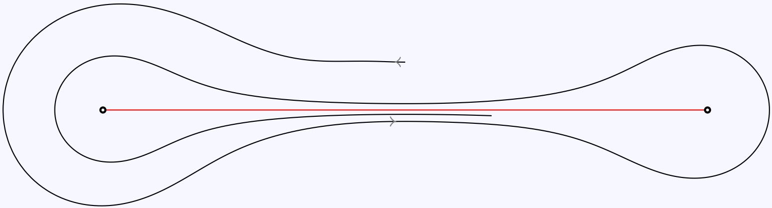

Figure 2: Counterexample on the plane minus two points.

- For the sphere (the plane and the projective plane), the lemma and the characterization are correct.
- However, we have found counterexamples for the lemma and for the proposed characterizations on proper open subsets of the plane and the projective plane.

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Characterization on open subsets of the sphere

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Open question

Characterize ω-limit sets on open subsets of the projective plane.

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Characterize ω-limit sets on open subsets of the projective plane.

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Characterize ω-limit sets on the Klein bottle and its open subsets.

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Characterize ω-limit sets on the torus (or, if possible, in surfaces in general) and on its open subsets.

3 Classification of unstable global attractors on the plane

Aim of the section

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We say that an orbit $\varphi(p)$ is stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $d_\infty(p, q) < \delta$ implies that all points from $\varphi(q)$ stay at a distance less than ϵ from $\varphi(p)$.

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AIM OF THE SECTION. To characterize, up to topological equivalence, the polynomial vector fields on $\mathbb{R}^2$ having an unstable global attractor.

First idea: to use the finite sectorial decomposition

Let Φ be an analytic flow on $\mathbb{R}^2$ and p be an isolated singular point. Then either is a center or there exists a neighbourhood of p which is a finite union of *hyperbolic*, *parabolic* and *elliptic* sectors.

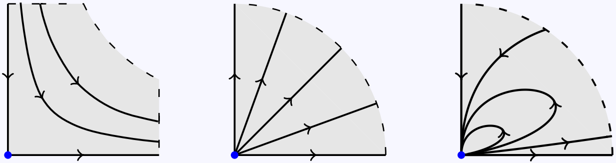

Figure 3: Hyperbolic, parabolic and elliptic sectors.

Necessary but not sufficient condition

Sharing the same sectorial decomposition is a necessary but not sufficient condition for being topologically equivalent.

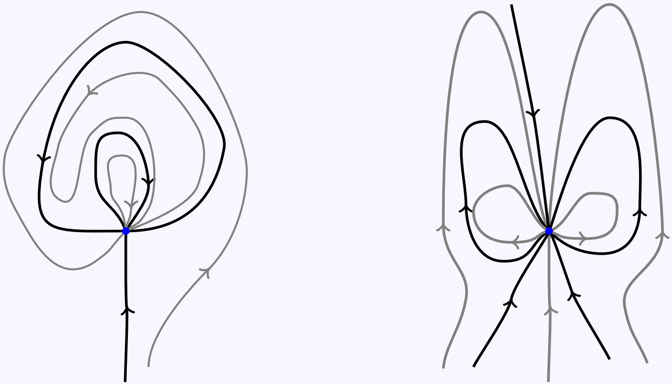

Figure 4: Two non-equivalent flows with the same sectorial decomposition (elliptic-parabolic-elliptic-parabolic-hyperbolic-parabolic-hyperbolic in counterclockwise sense).

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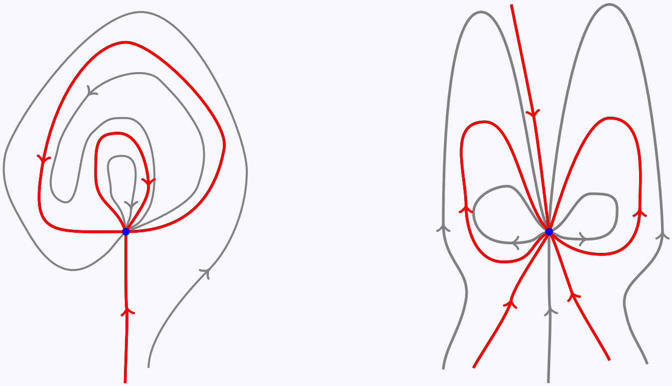

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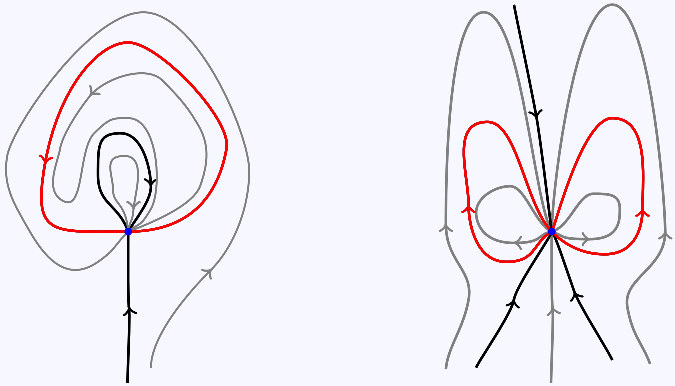

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The Markus-Neumann Theorem

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Let Φ be a continuous flow on $\mathbb{R}^2$.

Let Ω be an invariant region for Φ. We say that Ω is is parallel when the restriction of Φ to Ω is topologically equivalent to either the strip, the annular or the radial flow.

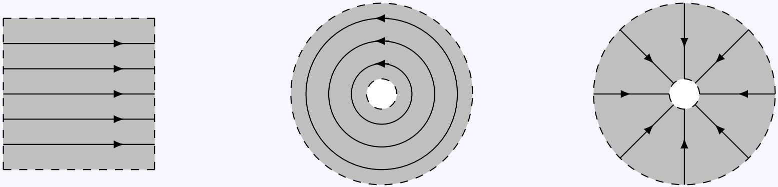

Figure 5: A strip, an annular and a radial region

The Markus-Neumann Theorem

An orbit $\varphi(p)$ of Φ is said to be ordinary if it is neighboured either by an annular region or by a strip or radial region Ω such that:

(i) $\alpha_\Phi(q) = \alpha_\Phi(p)$ and $\omega_\Phi(q) = \omega_\Phi(p)$ for any $q \in \Omega$;

(ii) $\mathrm{Bd}\,\Omega$ is the union of $\alpha_\Phi(p)$, $\omega_\Phi(p)$ and exactly two orbits $\varphi(a)$ and $\varphi(b)$ with $\alpha_\Phi(a) = \alpha_\Phi(b) = \alpha_\Phi(p)$ and $\omega_\Phi(a) = \omega_\Phi(b) = \omega_\Phi(p)$.

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Let us call $\mathcal{S}$ the union set of all its separatrices. The components of $\mathbb{R}^2 \setminus \mathcal{S}$ are called the canonical regions. By a separatrix configuration, $\mathcal{S}^+$, we mean the union of $\mathcal{S}$ together with a representative orbit from each canonical region.

The Markus-Neumann Theorem

Let Φ_1 and Φ_2 be two flows on $\mathbb{R}^2$ and let $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$ be, respectively, their separatrix configurations. We say that $\mathcal{S}_1^+$ and $\mathcal{S}_2^+$ are equivalent if there is a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$ carrying orbits of $\mathcal{S}_1^+$ onto orbits of $\mathcal{S}_2^+$ preserving time directions.

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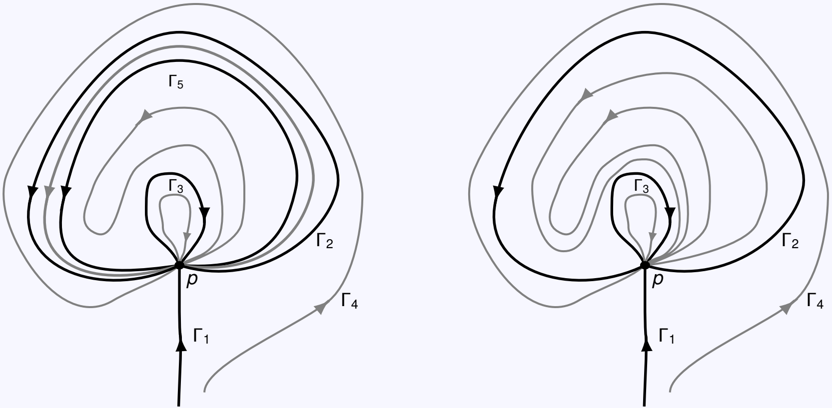
Markus-Neumann Theorem

Suppose that Φ_1 and Φ_2 are two continuous flows on $\mathbb{R}^2$ whose sets of singular points are discrete. Then Φ_1 and Φ_2 are topologically equivalent if and only if they have equivalent separatrix configurations.

L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math. Soc., 1954.

D. A. Neumann, *Classification of continuous flows on 2-manifolds*, Proc. Amer. Math. Soc., 1975.

Counterexample



$$\mathcal{S}_1 = \mathcal{S}_2 = \{p\} \cup \Gamma_1 \cup \Gamma_2$$
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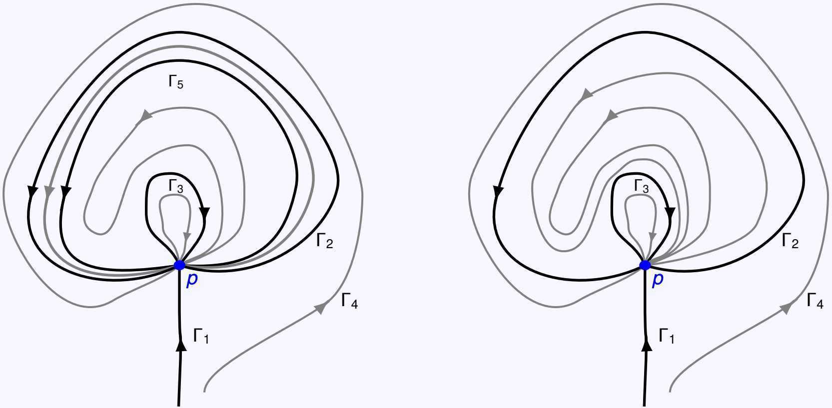

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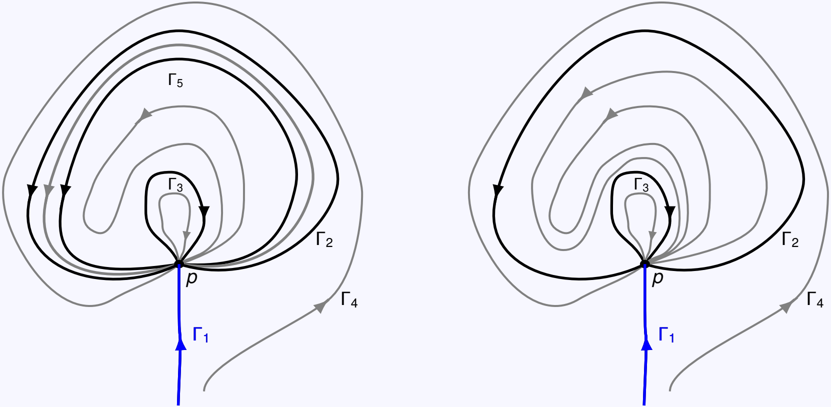

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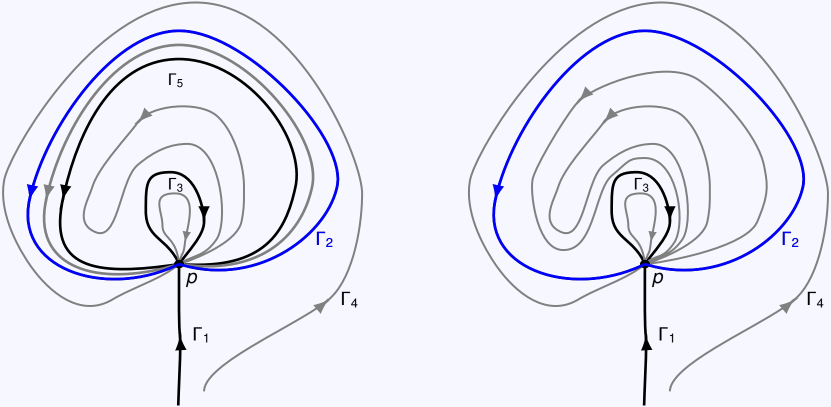

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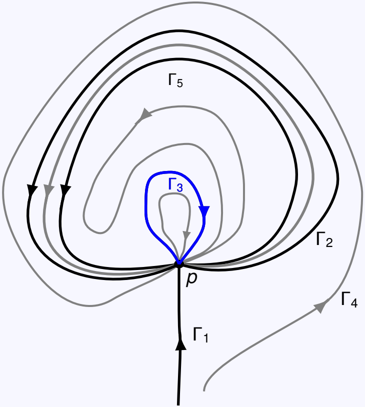

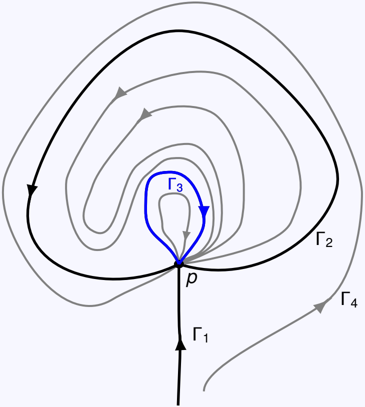

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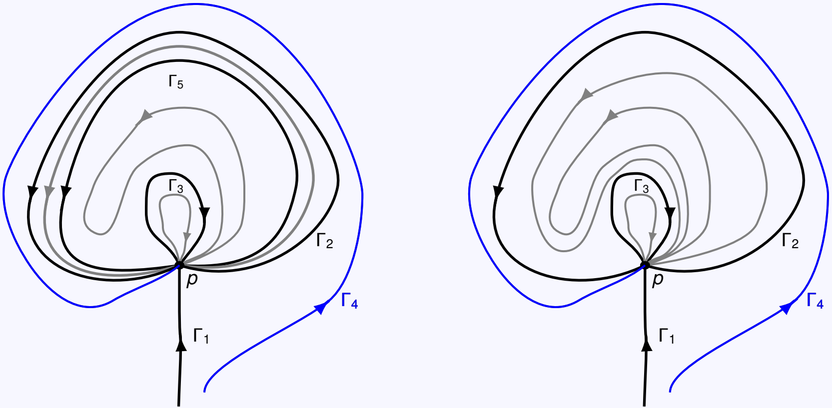

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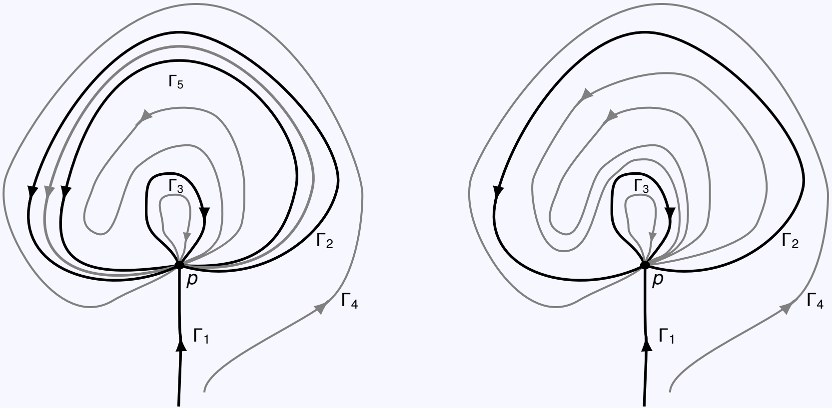

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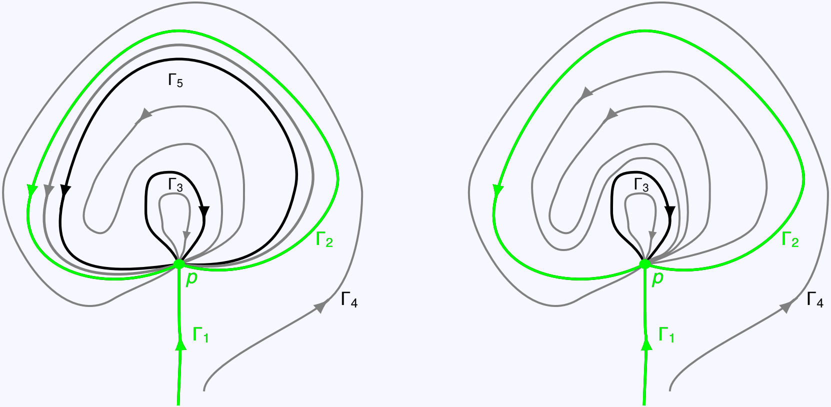

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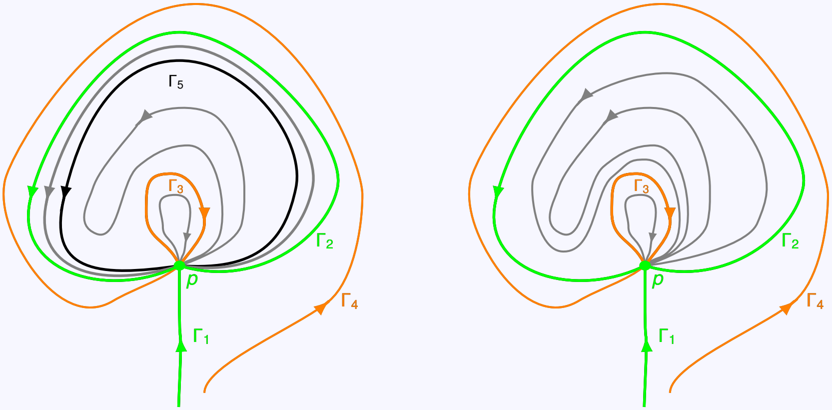

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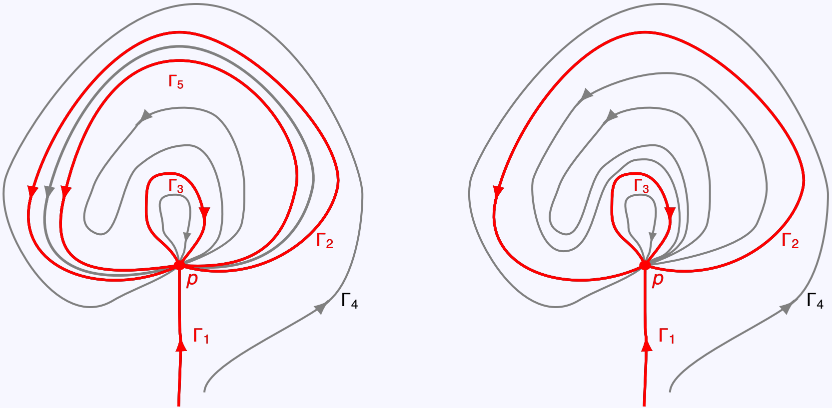

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A new formulation

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Let Φ_1 and Φ_2 be two continuous flows on $\mathbb{R}^2$ whose sets of singular points are discrete. Then Φ_1 and Φ_2 are topologically equivalent if and only if they have equivalent separator configurations.

J. G. E. and V. Jiménez, *On the Markus-Neumann Theorem*, to appear in J. Differential Equations, 2018. (arXiv:1707.05504).

Fortunatelly . . .

Books and papers invoking the Markus-Neumann theorem in the setting of analytic plane flows use an alternative definition of separatrix. Under the additional assumption of finiteness of singular points, an orbit is called a separatrix if and only if it is either a singular point, a limit cycle, or an orbit lying in the boundary of an hyperbolic sector.

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SO PAPERS IN THE ANALYTIC SETTINGS ARE CORRECT!

Characterization of unstable global attractors

J. G. E. and V. Jiménez, *A topological classification of plane polynomial systems having a globally attracting singular point*, Electron. J. Qual. Theory Differ. Equ., 2018. (arXiv:1708.00245)

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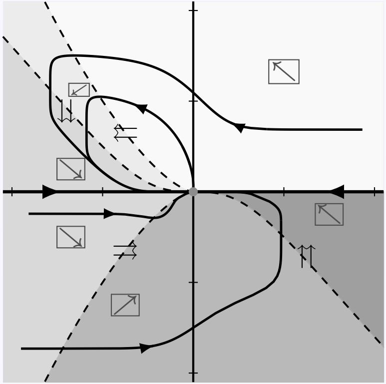

Figure 6: Phase portrait of $x' = -((1+x^2)y + x^3)^5$, $y' = y^2(y^2 + x^3)$.

4 Limit periodic sets

Aim of the section

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A limit periodic set for $(f_\lambda)_{\lambda\in\Lambda}$ at λ_0 is a closed set $\Gamma \subset \mathbb{R}^2$ for which there exist a sequence $(\lambda_n)_n$ and a sequence $(\gamma_n)_n$ of circles in $\mathbb{R}^2$ such that $(\lambda_n)_n$ converges to λ_0, $(\gamma_n)_n$ converges to $\hat{\Gamma}$ in the Hausdorff topology of $\mathbb{S}^2$ and, for every n, the vector field f_{λ_n} has γ_n as a limit cycle.

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AIM OF THE SECTION. To characterize, up to homeomorphism, the nature of limit periodic sets.

Example 1

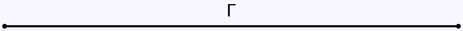

Figure 7: An arc.

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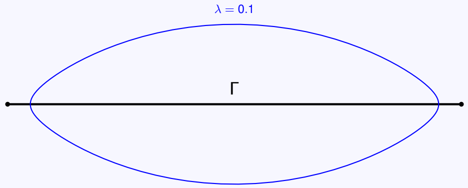

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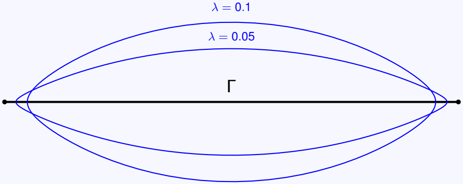

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Example 2

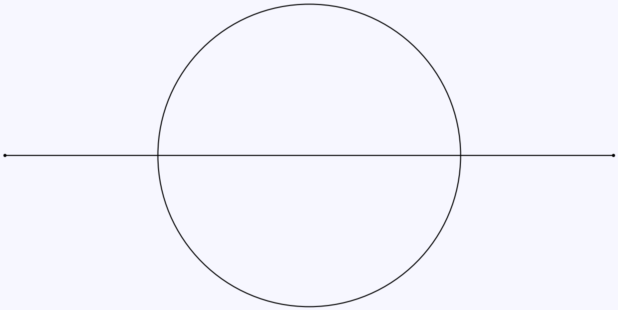

Figure 8: The union of an arc and a circle.

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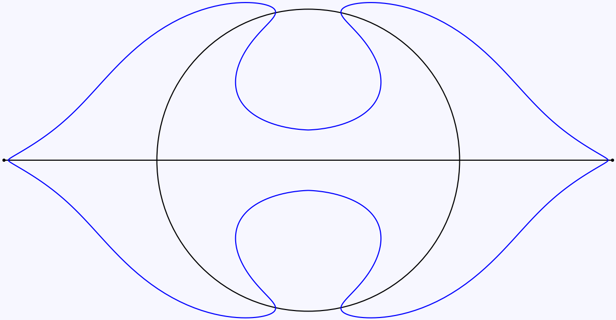

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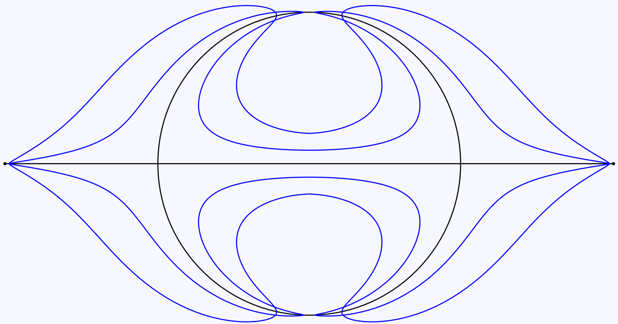

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Our solution

Theorem (A. Belotto and J. G. E.)

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a polynomial family of planar vector fields and Γ be a limit periodic set for $(f_\lambda)_{\lambda\in\Lambda}$. Then the compactification $\hat{\Gamma}\subset\mathbb{S}^2$ is a graph.

Conversely, if Γ is a nonempty closed subset of $\mathbb{R}^2$ whose compactification $\hat{\Gamma}\subset\mathbb{S}^2$ is a graph, then there exists a homeomorphism $h:\mathbb{S}^2\to\mathbb{S}^2$ and a polynomial family of planar vector fields $(f_\lambda)_{\lambda\in\Lambda}$ having $h(\Gamma)$ as a limit periodic set.

A. Belotto and J. G. E., *Topological classification of limit periodic sets of polynomial planar vector fields*, to appear in Publicacions Matemàtiques, 2018. (arXiv:1702.04965).

5 Minimal flows on nonorientable surfaces

Aim of the section

A flow on a surface S is called minimal if all the orbits are dense on S.

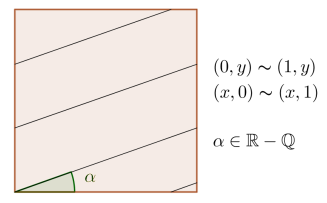

Figure 9: Irrational flow on the torus

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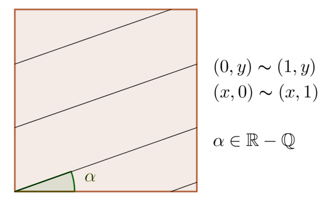

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AIM OF THE SECTION. To characterize surfaces admiting analytic minimal flows.

Preliminaries

- **The only compact surface admitting a minimal flow is the torus.** (A consequence of the Poincaré-Hopf Index Theorem).

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- G. Soler, in his Master Thesis (1999), proved that **a nonorientable surface admits transitive flows if and only if $g \geq 3$.**
- It is not difficult to show that **no nonorientable surface of genus $g = 3$ admits a minimal flow.**

Our results

Theorem (J. G. E., D. Peralta-Salas and G. Soler)

Let S be a nonorientable noncompact surface of finite genus g. Then there exists a minimal analytic flow on S if and only if $g \geq 4$.

Theorem (J. G. E., D. Peralta-Salas and G. Soler)

There exist nonorientable surfaces of infinite genus with minimal analytic flows.

J. G. E., D. Peralta-Salas and G. Soler, *Existence of minimal flows on nonorientable surfaces*, Discrete and Contin. Dyn. Syst., 2017. (arXiv:1608.08788).

Our results

Our proofs of both results consist in building surfaces and vector fields by suspending interval exchange transformations (a certain kind of piecewise affine maps of the unit interval).

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The keystone for the proof of the finite genus case is:

Theorem (Linero and Soler)

For every $n \geq 4$ and $1 \leq k \leq n$, there exists an (n, k)-i.e.t. all whose orbits are dense.

A. Linero and G. Soler, *Minimal interval exchange transformations with flips.*, Ergodic Theory of Dynamical Systems, 2017.

Our results

Our proofs of both results consist in building surfaces and vector fields by suspending interval exchange transformations (a certain kind of piecewise affine maps of the unit interval).

The proof of the infinite genus case is independent of the Linero and Soler Theorem:

Proposition (J. G. E., D. Peralta-Salas and G. Soler)

There exists a minimal i.e.t. with flips and with infinitely many discontinuities.

We conjecture that a future development in the study of interval exchange transformations will allow us to prove that any nonorientable surface of infinite genus is minimal.

MANY THANKS
FOR YOUR KIND ATTENTION!

Characterization on open subsets of the sphere

$\emptyset \subsetneq A \subsetneq \mathbb{S}^2$ is a shrub if it is a compact, connected, locally connected subset of $\mathbb{S}^2$ with connected complementary.

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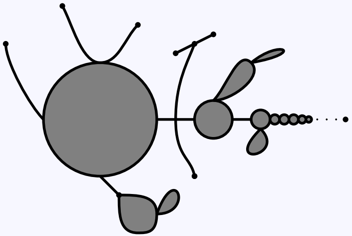
The closure of any of the components of $\mathrm{Int}\, A$ is a disk: the leaves of the shrub.



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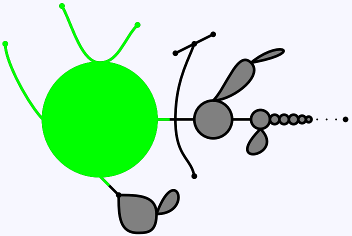

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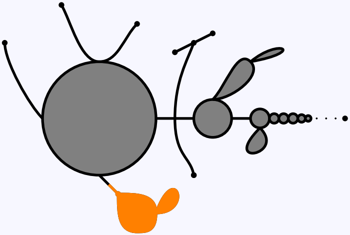

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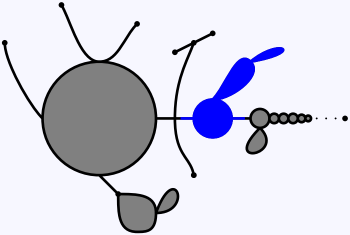

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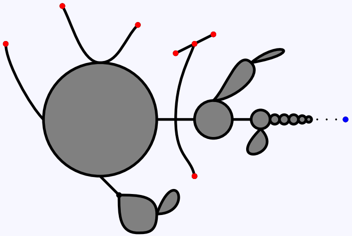
$u \in \mathrm{Bd}\, A$ is an odd vertex if either u is not a star point in $\mathrm{Bd}\, A$ or u is in no leaf of A and u is an n-star point in $\mathrm{Bd}\, A$ for some odd integer n.



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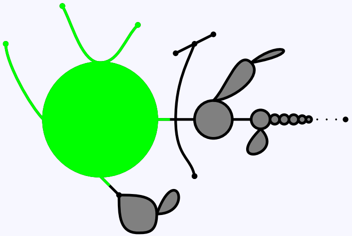

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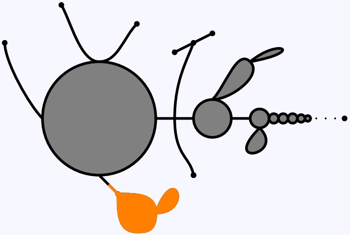

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The closure of any of the components of $\mathrm{Int}\, A$ is a disk: the leaves of the shrub.

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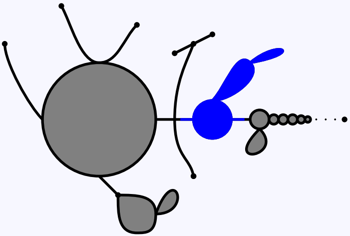

Figure 10: A shrub.

Characterization on open subsets of the sphere

Theorem

Let $T \subset \mathbb{S}^2$ be totally disconnected. If Φ is an analytic flow on $\mathbb{S}^2 \setminus T$, then any ω-limit set for Φ is the boundary of a shrub A. Moreover, all odd vertexes of the shrub are contained in T and every odd cactus in the shrub must intersect T.

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Theorem

Let $T \subset \mathbb{S}^2$ be totally disconnected. If Φ is an analytic flow on $\mathbb{S}^2 \setminus T$, then any ω-limit set for Φ is the boundary of a shrub A. Moreover, all odd vertexes of the shrub are contained in T and every odd cactus in the shrub must intersect T.

Conversely, let $A \subset \mathbb{S}^2$ be a shrub and let T contain all odd vertexes and one point from each of the odd cactuses of A. Then there are a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ and a C^∞ flow Φ on $\mathbb{S}^2$, analytic at least on $h(\mathbb{S}^2 \setminus T)$, having the boundary of $h(A)$ as an ω-limit set.

J. G. E. and V. Jiménez, *A topological characterization of the omega-limit sets of analytic vector fields on open subsets of the sphere*, to appear in Discrete Contin. Dyn. Syst. Ser. B. (arXiv:1711.00567).