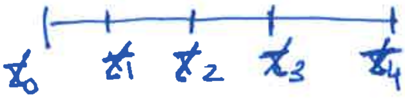
Tema 6: Resolución Numérica de EDO

1.

$$\left.\begin{array}{ll} u'(t) = -3u(t) - t^2 & t \geq 0 \\ u(0) = 3 & \end{array}\right\}$$

a) $I = [0, 3]$ $\quad n = 4$



4 subintervalos

$$h = \frac{3-0}{4} = 3/4 = 0.75$$

$$t_0 = 0 \Rightarrow [0, 3/4, 6/4, 9/4, 12/4] = [0, 0.75, 1.5, 2.25, 3]$$

Euler Explícito:

$$u_{n+1} = u_n + h \cdot f(t_n, u_n) = u_n + h \cdot (-3u_n - t_n^2)$$

$$\boxed{u_{n+1} = u_n \cdot (1 - 3h) - h \cdot t_n^2}$$

$$u_1 = u_0(1 - 3 \cdot h) - h \cdot t_0^2 = 3 \cdot (1 - 3 \cdot 3/4) - 3/4 \cdot 0^2 = -\frac{15}{4}$$

$$u_2 = u_1(1 - 3h) - h \cdot t_1^2 = -\frac{15}{4}(1 - 9/4) - 3/4 \cdot (3/4)^2 = \frac{273}{64} \approx 4.265625$$

$$u_3 = u_2(1 - 3h) - h t_2^2 = \frac{273}{64} \cdot (1 - 9/4) - 3/4 \cdot \left(\frac{6}{4}\right)^2 = \frac{-1797}{256} \simeq -7.01953125$$

$$u_4 = u_3(1 - 3h) - h t_3^2 = \frac{-1797}{256}\left(1 - \frac{9}{4}\right) - \frac{3}{4}\left(\frac{9}{4}\right)^2 = \frac{5097}{1024} \simeq 4.9775390625$$

b) Es una edo lineal con parámetro $a=-3$. Para garantizar estabilidad debe cumplirse

$$\Delta t < \frac{2}{|a|}$$

pero $\Delta t = 0.75$ y $\frac{2}{|a|} = \frac{2}{3} \approx 0.\widehat{6} \Rightarrow 0.75 > 0.\tilde{6}$ es inestable.

c) Podemos tomar cualquier Δt que cumpla la condición anterior; por tanto

$$\Delta t < 2/3 = 0.\tilde{6}\ldots$$

Así podríamos tomar, por ejemplo, $h = 0.6 = \frac{3}{5}$ o $h = 0.5 = \frac{1}{2} = \frac{3}{6}$ es decir, podemos tomar 5 o 6 subintervalos. Tomaremos 5, que necesitamos menos cálculos. $t \in [0, 3/5, 6/5, 9/5, 12/5, 15/5]$

$$u_{n+1} = u_n + h\cdot f(t_n, u_n) = u_n + \frac{3}{5}\cdot(-3u_n - t_n^2) \Rightarrow$$

$$u_{n+1} = u_n\cdot(1 - 9/5) - \frac{3}{5}t_n^2 = -\frac{4}{5}u_n - \frac{3}{5}\cdot t_n^2$$

$$u_1 = -\frac{4}{5}u_0 - \frac{3}{5}t_0^2 = -\frac{4}{5}\cdot 3 - \frac{3}{5}\cdot 0^2 = -\frac{12}{5} \approx -2.4.$$

$$u_2 = -\frac{4}{5}\cdot u_1 - \frac{3}{5}t_1^2 = -\frac{4}{5}\cdot\left(-\frac{12}{5}\right) - \frac{3}{5}\cdot\left(\frac{3}{5}\right)^2 = \frac{213}{125} \approx 1.704$$

$$u_3 = -\frac{4}{5}u_2 - \frac{3}{5}t_2^2 = -\frac{4}{5}\cdot\left(\frac{213}{125}\right) - \frac{3}{5}\left(6/5\right)^2 = -\frac{1392}{625} \approx -2.2272$$

$$u_4 = -\frac{4}{5}u_3 - \frac{3}{5}\cdot t_3^2 = -\frac{4}{5}\left(-\frac{987}{625}\right)\cdot\frac{3}{5}\left(\frac{9}{5}\right)^2 = -\frac{507}{3125} \approx -0.16224.$$

$$u_5 = -\frac{4}{5}u_4 - \frac{3}{5}t_4^2 = -\frac{4}{5}\left(\frac{3273}{3125}\right) - \frac{3}{5}\left(\frac{12}{5}\right) = -\frac{51972}{15625} \approx -3.326208$$

d) $h = \Delta t = \frac{3}{4}$ $\qquad t \in \{0, 3/4, 6/4, 9/4, 12/4\}$

Euler Implícito: $\quad u_{n+1} = u_n + \Delta t \cdot f(t_{n+1}, u_{n+1})$

$$u_{n+1} = u_n + \frac{3}{4} \cdot \left[-3u_{n+1} - t_{n+1}^2\right] \Rightarrow$$

$$u_{n+1} = u_n - \frac{9}{4}u_{n+1} - \frac{3}{4}t_{n+1}^2 \Rightarrow \left(1 + \frac{9}{4}\right)u_{n+1} = u_n - \frac{3}{4}t_{n+1}^2$$

$$\frac{13}{4}u_{n+1} = u_n - \frac{3}{4}t_{n+1}^2 \Rightarrow u_{n+1} = \frac{4}{13} \cdot \left[u_n - \frac{3}{4} \cdot t_{n+1}^2\right]$$

$$u_1 = \frac{4}{13} \cdot \left[u_0 - \frac{3}{4} \cdot t_1^2\right] = \frac{4}{13} \cdot \left[3 - \frac{3}{4} \cdot \left(\frac{3}{4}\right)^2\right] = \frac{165}{208} = 0.7932692\ldots$$

$$u_2 = \frac{4}{13} \cdot \left[u_1 - \frac{3}{4} \cdot t_2^2\right] = \frac{4}{13} \cdot \left[\frac{165}{208} - \frac{3}{4} \cdot \left(\frac{6}{4}\right)^2\right] = \frac{-93}{338} = -0.2751479\ldots$$

$$u_3 = \frac{4}{13} \cdot \left[u_2 - \frac{3}{4} t_3^2\right] = \frac{4}{13} \cdot \left[\frac{-93}{338} - \frac{3}{4} \cdot \left(\frac{9}{4}\right)^2\right] = \frac{-44043}{35152} \simeq -1.2529301\ldots$$

$$u_4 = \frac{4}{13} \cdot \left[u_3 - \frac{3}{4} \cdot t_4^2\right] = \frac{4}{13} \cdot \left[\frac{-44043}{35152} - \frac{3}{4} \cdot \left(\frac{12}{4}\right)^2\right] = \frac{-281319}{114244} \simeq -2.462440$$

2.—

$$\left.\begin{aligned} u' &= -3u^2 - t^2 \qquad t \geq 0 \\ u(0) &= 3 \end{aligned}\right\}$$

a) $\Delta t = 3/4 \qquad t \in \{0, 3/4, 6/4, 9/4, 12/4 = 3\}$

Explícito: $u_{n+1} = u_n + \Delta t \cdot f(t_n, u_n)$

$$u_{n+1} = u_n + \frac{3}{4}\cdot\left(-3u_n^2 - t_n^2\right) = u_n - \frac{9}{4}u_n^2 - \frac{3}{4}t_n^2$$

$$u_1 = u_0 - \frac{9}{4}u_0^2 - \frac{3}{4}\cdot t_0^2 = 3 - \frac{9}{4}(3)^2 - \frac{3}{4}\cdot(0)^2 = -\frac{69}{4} = -17.25$$

$$u_2 = u_1 - \frac{9}{4}u_1^2 - \frac{3}{4}t_1^2 = -\frac{69}{4} - \frac{9}{4}\left(-\frac{69}{4}\right)^2 - \frac{3}{4}\left(\frac{3}{4}\right)^2 = -\frac{10995}{16} = -687.1875$$

$$u_3 = u_2 - \frac{9}{4}u_2^2 - \frac{3}{4}t_2^2 = -1063198.86\ldots \approx -10^6.$$

$$u_4 = u_3 - \frac{9}{4}u_3^2 - 3/4\, t_3^2 = -2.54 \times 10^{12}$$

claramente inestable.

b) Para garantizar la estabilidad debería cumplirse en cada paso la siguiente desigualdad

$$\Delta t < \frac{2}{\left|\frac{\partial f}{\partial u}(t_n, u_n)\right|}$$

Con $\frac{\partial f}{\partial u} = -6 \cdot u \Rightarrow \frac{\partial f}{\partial u}(t_n, u_n) = -6u_n$; por tanto

$$\Delta t < \frac{2}{|-6u_n|} = \frac{2}{6} \cdot \frac{1}{|u_n|} = \frac{1}{3|u_n|}.$$

- En el 1er paso $u_0 = 3 \Rightarrow \Delta t < \frac{1}{3 \cdot |3|} = \frac{1}{9} = 0.\hat{1}$

pero hemos tomado $\Delta t = 3/4$, luego la desigualdad no se cumple.

- Para el 2º paso y suponiendo $u_1 = -17.25 \Rightarrow$

$$\frac{1}{3|u_n|} = \frac{1}{3 \cdot 17.25} \cong 0.02.$$

Bastante menor que 0.75.

c) $h = \Delta t = 3/4 \qquad t \in \{0, 3/4, 6/4, 9/4, 3\}$

Euler implícito: $u_{n+1} = u_n + h \cdot f(t_{n+1}, u_{n+1})$.

$$u_{n+1} = u_n + \frac{3}{4} \cdot \left(-3u_{n+1}^2 - t_{n+1}^2\right)$$

$$u_{n+1} = u_n - \frac{9}{4} u_{n+1}^2 - \frac{3}{4} t_{n+1}^2 \quad \Rightarrow$$

$$\frac{9u_{n+1}^2}{4} + u_{n+1} + \left(3/4\, t_{n+1}^2 - u_n\right) = 0$$

$$\boxed{9u_{n+1}^2 + 4u_{n+1} + \left(3t_{n+1}^2 - 4u_n\right) = 0}$$

Ecuación de 2º grado que podemos resolver utilizando la fórmula correspondiente:

$$u_{n+1} = \frac{-4 \pm \sqrt{16 - 36\left(3t_{n+1}^2 - 4u_n\right)}}{18} =$$

$$= \frac{-4 \pm \sqrt{16 - 36\left(3t_{n+1}^2 - 4u_n\right)}}{18} =$$

$$= \frac{-4 \pm 2\sqrt{4 - 9\left(3t_{n+1}^2 - 4u_n\right)}}{18}$$

$$\boxed{u_{n+1} = \frac{-2 \pm \sqrt{4 - 9\left(3t_{n+1}^2 - 4u_n\right)}}{9}}$$

Para

$$u_1 = \frac{-2 \pm \sqrt{4 - 9\left(3\cdot\left(3/4\right)^2 - 4\cdot 3\right)}}{9} = \frac{-8 \pm \sqrt{1549}}{36}$$

Tenemos 2 opciones.

$$u_1^1 = \frac{-8 + \sqrt{1549}}{36} = 0.871037\ldots$$

$$u_1^2 = \frac{-8 - \sqrt{1549}}{36} \simeq -1.315482.$$

Como $u_0 = 3$, tenemos que elegir aquella solución que esté más cerca de este valor, es decir

$$\boxed{u_1 = 0.871037}.$$

$$u_2 = \frac{-2 \pm \sqrt{4 - 9\left(3t_2^2 - 4u_1\right)}}{9} =$$

$$= \frac{-2 \pm \sqrt{4 - 9\cdot\left(3\left(6/4\right)^2 - 4\cdot 0.871037\right)}}{9}$$

$$= \frac{-2 \pm \sqrt{4 - 9\cdot\left(\frac{27}{4} - 3.484148\right)}}{9} = \frac{-2 \pm \sqrt{4 - 29.392668}}{9} \notin \mathbb{R}$$

No se puede aplicar.

3.—

$$\begin{cases} u''(t) + 2u'(t) + u(t) = -t^2 & t \geq 0 \quad (0) \\ u(0) = 3 \\ u'(0) = 0 \end{cases}$$

a) Hacemos cambio

$$x(t) = u(t) \quad (1)$$

$$y(t) = u'(t) \quad (2)$$

Por tanto, derivando en ambas ecuaciones.

$$\left.\begin{aligned} x'(t) &= \underbrace{u'(t) =}_{(2)} y(t) \\ y'(t) &= \underbrace{u''(t) =}_{(0)} -2u'(t) - u(t) - t^2 = -2y(t) - x(t) - t^2 \end{aligned}\right\}$$

Y el sistema queda.

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -t^2 \end{pmatrix}$$

b) Euler explícito: "$u_{n+1} = u_n + \Delta t \cdot f(t_n, u_n)$".

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \Delta t \cdot \left[\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ -t_n^2 \end{pmatrix} \right]$$

$I=[0,4] \quad n=4 \Rightarrow \quad \Delta t = \frac{4-0}{4} = 1 \; . \Rightarrow \; t = \{0,1,2,3,4\}$

La ecuación queda.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ -t_n^2 \end{pmatrix}$$

$$= \begin{pmatrix} x_n + y_n \\ -x_n - y_n - t_n^2 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 + y_0 \\ -x_0 - y_0 - t_0^2 \end{pmatrix} = \begin{pmatrix} 3+0 \\ -3-0-0^2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ -x_1 - y_1 - t_1^2 \end{pmatrix} = \begin{pmatrix} 3-3 \\ -3-(-3)-1^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 + y_2 \\ -(x_2+y_2) - t_2^2 \end{pmatrix} = \begin{pmatrix} 0+(-1) \\ -(0+(-1))-2^2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_3 + y_3 \\ -(x_3+y_3) - t_3^2 \end{pmatrix} = \begin{pmatrix} -1+(-3) \\ -(-1+(-3))-3^2 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

c) Es un sistema de EDO LINEAL. por tanto la condición de estabilidad es

in

$$\Delta t < \frac{2}{\max\{|\lambda_j| \mid \lambda_j \text{ valor propio de } A\}}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda)+1$$

$$p(\lambda) = 2\lambda + \lambda^2 + 1 = (\lambda+1)^2 \Rightarrow \lambda_1 = \lambda_2 = -1 \quad \text{valores propios}$$

$$|\lambda_1| = |\lambda_2| = 1 \Rightarrow \max\{|\lambda_j| \mid \lambda_j \text{ valor propio de } A\} = 1.$$

$$\Delta t = 1 < \frac{2}{1} = 2$$

MÉTODO ESTABLE PARA ESTE VALOR DE Δt.

d) EULER IMPLÍCITO : " $u_{n+1} = u_n + h \cdot f(t_{n+1}, u_{n+1})$ "

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \Delta t \cdot \left[\begin{pmatrix} y_{n+1} \\ -x_{n+1} - 2y_{n+1} - t_{n+1}^2 \end{pmatrix} \right]$$

Para $\Delta t = 1$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} y_{n+1} \\ -x_{n+1} - 2y_{n+1} - t_{n+1}^2 \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} - \begin{pmatrix} y_{n+1} \\ -x_{n+1} - 2y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ -t_{n+1}^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n - t_{n+1}^2 \end{pmatrix}$$

Calculamos inversa de la matriz

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \quad Adj(B^T) = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \quad \det(B) = 4$$

$$B^{-1} = \begin{pmatrix} 3/4 & 1/4 \\ -1/4 & 1/4 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

Por tanto

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n - t_{n+1}^2 \end{pmatrix} =$$

$$= \frac{1}{4} \cdot \begin{pmatrix} 3x_n + y_n - t_{n+1}^2 \\ -x_n + y_n - t_{n+1}^2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3x_0 + y_0 - t_1^2 \\ -x_0 + y_0 - t_1^2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 \cdot 3 + 0 - 1^2 \\ -3 + 0 - 1^2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3x_1 + y_1 - t_2^2 \\ -x_1 + y_1 - t_2^2 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3 \cdot 2 + (-1) - 2^2 \\ -2 + (-1) - 2^2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -7/4 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3x_2 + y_2 - t_3^2 \\ -x_2 + y_2 - t_3^2 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3 \cdot \frac{1}{4} + (-7/4) - 3^2 \\ -\frac{1}{4} - \frac{7}{4} - 3^2 \end{pmatrix} = \begin{pmatrix} -\frac{10}{4} \\ -11/4 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3x_3 + y_3 - t_4^2 \\ -x_3 + y_3 - t_4^2 \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 3 \cdot (-\frac{10}{4}) + (-\frac{11}{4}) - 4^2 \\ -(-\frac{10}{4}) + (-\frac{11}{4}) - 4^2 \end{pmatrix} = \begin{pmatrix} -\frac{105}{16} \\ -\frac{65}{16} \end{pmatrix}$$