The Liouville equation in a half-plane

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Abstract

We classify the solutions of the equation $\Delta u + ae^u = 0$ in the half-plane $\mathbb{R}^2_+$ that satisfy the Neumann boundary condition $\partial u/\partial t = c\, e^{u/2}$ on $\partial\mathbb{R}^2_+$. An analogous problem in the once punctured disc $\mathbb{D}^* \subset \mathbb{R}^2$ is also solved.

1 Introduction

In this paper we classify all the solutions $u \in C^2(\overline{\mathbb{R}^2_+})$ of the following nonlinear boundary value problem:

$$\begin{cases} \Delta u + ae^u = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : t > 0\}, \\ \dfrac{\partial u}{\partial t} = ce^{u/2} & \text{on } \partial\mathbb{R}^2_+, \quad a, c \in \mathbb{R}. \end{cases} \tag{1.1}$$

The *Liouville equation* $\Delta u + ae^u = 0$ is a famous quasilinear elliptic PDE that traces back to pioneer works by Liouville, Poincaré and Picard among others, and that has a clear geometric interpretation. Indeed, if u is a solution to the Liouville equation, then e^u is the conformal factor that turns the flat metric $ds^2 + dt^2$ into a metric of constant curvature $a/2$. On the other hand, the boundary condition $\partial u/\partial t = ce^{u/2}$ simply means that the geodesic curvature of the real axis with respect to the constant curvature metric $e^u(ds^2 + dt^2)$ is constant of value $-c/2$.

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In [Zha] L. Zhang classified the solutions to (1.1) under the additional assumption

$$\int_{\mathbb{R}^2_+} e^u < +\infty, \tag{1.2}$$

and found out the pairs (a, c) for which the problem (1.1) has a solution under the hypothesis (1.2). Previous results on the problem (1.1) under integral finiteness assumptions had been obtained in [LiZh, Ou] (see also [ChLi, ChWa, HaWa]). The case in which the solution presents a conical singularity at a certain boundary point has been recently solved in [JWZ], again under the hypothesis (1.2). More generally, the study of the solutions to Neumann boundary problems in a half-space for semilinear elliptic equations is currently a topic of interest (see for instance, among others, [LiZha, CaSo] and references therein).

Our aim in this paper is to generalize Zhang's theorem, by finding all the solutions to (1.1) without any integral finiteness hypothesis.

There is another source of motivation for the problem we are considering here. In dimension $n \geq 3$, the problem analogous to (1.1) is

$$\begin{cases} \Delta u + a u^{\frac{n+2}{n-2}} = 0, \ u > 0, & \text{in } \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}, \\ \dfrac{\partial u}{\partial x_n} = c u^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+, \quad a, c \in \mathbb{R}. \end{cases} \tag{1.3}$$

This problem was completely solved for $a \geq 0$ by Y.Y. Li and M. Zhu in [LiZh] (see [CSF] for the solution in the $a < 0$ case). So, our description of all the solutions to (1.1) settles the missing two dimensional case in the above classification.

Our classification of all the solution to (1.1), which takes advantage of the connection of the Liouville equation with complex analysis, will be exposed in Section 2 and proved in Section 3. In Section 4 we shall use this classification theorem to describe all the solutions $u \in C^2(\mathbb{D}^* \cup \mathbb{S}^1)$ to

$$\begin{cases} \Delta u + a e^u = 0 & \text{in } \mathbb{D}^* = \{z \in \mathbb{R}^2 \equiv \mathbb{C} : 0 < |z| < 1\}, \\ \dfrac{\partial u}{\partial \nu} = c e^{u/2} + 2 & \text{on } \mathbb{S}^1 = \{z : |z| = 1\}, \quad a, c \in \mathbb{R}, \end{cases} \tag{1.4}$$

where ν denotes the interior unit normal of $\mathbb{S}^1$. This problem was solved in [HaWa] for the non-punctured unit disk $\mathbb{D}$, i.e. when $u \in C^2(\mathbb{D})$. Again, the Neumann boundary condition in (1.4) translates geometrically into the fact that the geodesic curvature of the boundary with respect to the metric $e^u(ds^2 + dt^2)$ is constant of value $-c/2$. In the end, we will show that any solution to (1.4) such that

$$\int_{\mathbb{D}^*} e^u < +\infty$$

and which does not extend smoothly across the origin is radially symmetric, and we will describe all such solutions explicitly.

Let us finally point out that our approach to (1.1) also provides alternative proofs of previous results on the problem (1.1)-(1.2), which were originally obtained by moving planes or moving spheres arguments (see Remark 4, Remark 11 and Corollary 5). Besides, our classification might also be useful to investigate the problem (1.1) under some decay conditions at infinity weaker than (1.2), by means of the theory of entire holomorphic functions.

2 The classification theorem

Let us start by recalling the following classical result, mainly due by Liouville [Lio] (see also [Bry, ChWa]).

Theorem 1 *Let* $u : \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{R}$ *denote a solution to* $\Delta u + ae^u = 0$ *in a simply connected domain* Ω*. Then there exists a locally univalent meromorphic function* g *(holomorphic with* $2 + a|g|^2 > 0$ *if* $a \leq 0$*) in* Ω *such that*

$$u = \log \frac{4|g'|^2}{(1 + \frac{a}{2}|g|^2)^2}. \tag{2.1}$$

Conversely, if g *is a locally univalent meromorphic function (holomorphic with* $2 + a|g|^2 > 0$ *if* $a \leq 0$*) in* Ω*, then* (2.1) *is a solution to* $\Delta u + ae^u = 0$ *in* Ω*.*

Let us point out that, up to a dilation, we may choose $a = 2\varepsilon$, $\varepsilon \in \{-1, 0, 1\}$, in the equation $\Delta u + ae^u = 0$. This will be assumed from now on. It is also interesting to point out that the function g in the above theorem is unique up to a transformation of the form

$$g \mapsto \frac{\alpha g - \bar{\beta}}{\varepsilon \beta g + \bar{\alpha}}, \qquad |\alpha|^2 - \varepsilon|\beta|^2 = 1. \tag{2.2}$$

In order to expose the solution to (1.1), let us define $\mathcal{A}$ as the set of all real analytic functions $h(s) : \mathbb{R} \to \mathbb{R}$ whose convergence radius is infinite. This indicates that any $h(s) \in \mathcal{A}$ extends to an entire function $h(z)$, and that the class $\mathcal{A}$ consists of the restriction to the real line $\mathbb{R}$ of all entire functions in $\mathbb{C}$ taking real values on $\mathbb{R}$.

Once here, we can state the description of all solutions to (1.1). We will split it into three separate cases depending on the sign of a. From now on we shall identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{C}_+$ with $\mathbb{R}^2_+$.

The solution for $\Delta u + 2e^u = 0$

Let $h(s) \in \mathcal{A}$ such that $h(s) \neq 0$ at every point, and assume that the entire extension $h(z)$ only has simple zeros with $h'(z_0) = \pm i$ at each of them. Then the map

$$u = \log \frac{4|g'(z)|^2}{(1 + |g(z)|^2)^2}, \tag{2.3}$$

where

$$g(z) = \frac{2\,\mathrm{sg(c)}}{|c| - \sqrt{c^2+4}} \exp\left(i \int_{s_0}^{z} \frac{1}{h(w)} dw \right), \qquad s_0 \in \mathbb{R} \tag{2.4}$$

is a solution to (1.1) for $a = 2$ and $c \in \mathbb{R}$. And conversely, any solution to (1.1) for $a = 2$ and $c \in \mathbb{R}$ is of this form. Here, sg(c)=1 if $c \geq 0$ and sg(c)= -1 if $c < 0$.

The solution for $\Delta u = 0$

If $c = 0$, a function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is a solution to (1.1) for $a = 0$ if and only if $u = \mathrm{Re}\, h(z)$, where $h(z)$ is the entire extension of a function $h(s) \in \mathcal{A}$.

Assume now that $c \neq 0$. Let $h(s) \in \mathcal{A}$ such that $h(s) \neq 0$ at every point, and assume that the entire extension $h(z)$ only has simple zeros with $h'(z_0) = i$ at each of the zeros lying in $\mathbb{C}_+$. Then the map $u = \log 4|g'(z)|^2$ where

$$g(z) = -\frac{2}{c} \exp\left(i \int_{s_0}^{z} \frac{1}{h(w)} dw \right), \qquad s_0 \in \mathbb{R}$$

is a solution to (1.1) for $a = 0$ and $c \neq 0$. Conversely, all solutions to (1.1) for $a = 0$ and $c \neq 0$ are constructed in this way.

The solution for $\Delta u - 2e^u = 0$

The case $c > -2$**:** In this situation, the problem (1.1) for $a = -2$ does not have a solution. This was proved for $c \geq 0$ in [Zha].

The case $c = -2$**:** The only solutions to the problem (1.1) for $a = -2$ and $c = -2$ are given by

$$u(s,t) = \log\left(\frac{h_0^2}{(1 + h_0 t)^2} \right)$$

for any positive real constant h_0.

The case $c < -2$**:** Let $h(s) \in \mathcal{A}$ such that $h(s) \neq 0$ at every point. Suppose that the entire extension $h(z)$ only has simple zeros with $h'(z_0) = i$ at each of the zeros lying in $\mathbb{C}_+$, and that the function

$$\mathrm{Im} \int_{s_0}^{z} \frac{1}{h(w)} dw, \qquad s_0 \in \mathbb{R} \tag{2.5}$$

is bounded from below in $\mathbb{C}_+$. Then the map

$$u = \log \frac{4|g'(z)|^2}{(1 - |g(z)|^2)^2}, \tag{2.6}$$

where

$$g(z) = \frac{2}{-c + \sqrt{c^2-4}} \exp\left(i \int_{s_0}^{z} \frac{1}{h(w)} dw \right), \qquad s_0 \in \mathbb{R} \tag{2.7}$$

is a solution to (1.1) for $a = -2$ and $c < -2$. Conversely, any solution to (1.1) for $a = -2$ and $c < -2$ is of this form.

3 Proof of the classification theorem

Let $u : \overline{\mathbb{R}^2_+} \equiv \mathbb{C}_+ \cup \mathbb{R} \to \mathbb{R}$ denote a solution to (1.1), where we are assuming without loss of generality that $a = 2\varepsilon$, $\varepsilon \in \{-1, 0, 1\}$, and consider the meromorphic function $g : \mathbb{C}_+ \to \mathbb{C} \cup \{\infty\}$ given by Theorem 1. In addition, let us recall the Wirtinger operators associated to a complex parameter $z = s + it$ of a complex domain $\Omega \subset \mathbb{C}$:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right).$$

Then, by differentiating (2.1) we easily obtain that

$$u_{zz} - \frac{1}{2}\,u_z^2 = \{g, z\} := \left(\frac{g_{zz}}{g_z}\right)_z - \frac{1}{2}\left(\frac{g_{zz}}{g_z}\right)^2. \tag{3.1}$$

Here $\{g, z\}$ is the classical *Schwarzian derivative* of the meromorphic function g with respect to z. This shows that

$$Q dz^2 := (u_{zz} - \frac{1}{2}\,u_z^2) dz^2$$

is a holomorphic quadratic differential in $\mathbb{C}_+$. Moreover, Q extends continuously to $\mathbb{R}$, and by the condition $\partial u / \partial t = c e^{u/2}$ in (1.1) we see that

$$\operatorname{Im} Q(s, 0) = -\frac{1}{2}\left(\frac{c}{2} u'(s) e^{u(s)/2} - \frac{c}{2} u'(s) e^{u(s)/2}\right) = 0, \tag{3.2}$$

i.e. Q takes real values when restricted to $\mathbb{R}$. Therefore, by the Schwarz reflection principle, the holomorphic function $Q = u_{zz} - u_z^2/2$ extends to an entire function by means of $Q(\bar{z}) = \overline{Q(z)}$.

Let us recall at this point the following classical fact: if $q(z)$ is a holomorphic function in a simply connected domain, then the equation $\{g, z\} = q(z)$ always has a locally univalent meromorphic solution g, which is unique up to linear fractional transformations. In our case, we have $\{g, z\} = Q$ in $\mathbb{C}_+$. And since Q can be extended to the complex plane we can assure that g can also be extended to a locally univalent meromorphic map globally defined in $\mathbb{C}$.

Once here, let $\mathcal{Q}(\varepsilon)$ denote the 2-dimensional model space of constant curvature $\varepsilon \in \{-1, 0, 1\}$, that is, $\mathcal{Q}(0) = \mathbb{R}^2$ and

$$\mathcal{Q}(\varepsilon) = \begin{cases} \mathbb{S}^2 = \left\{(x_0, x_1, x_2) : x_0^2 + x_1^2 + x_2^2 = 1\right\} & \text{if } \varepsilon = 1, \\ \\ \mathbb{H}^2 = \left\{(x_0, x_1, x_2) : -x_0^2 + x_1^2 + x_2^2 = -1,\ x_0 > 0\right\} & \text{if } \varepsilon = -1. \end{cases}$$

In addition, let us consider the *stereographic projection* π of $\mathcal{Q}(\varepsilon)$ into $\mathbb{C} \cup \{\infty\}$, defined by

$$\pi(x_0, x_1, x_2) = \frac{x_1 + i x_2}{1 - \varepsilon x_0} \qquad \text{if } \varepsilon = \pm 1,$$

and $\pi(x_1,x_2)=x_1+ix_2$ if $\varepsilon=0$. With this, one of the key points of our solution to (1.1) is the following result.

Theorem 2 *Let $f(s):I\subset\mathbb{R}\to\mathbb{R}$ denote a real analytic function, and $\varepsilon\in\{-1,0,1\}$. The unique solution to the Cauchy problem*

$$\left\{\begin{array}{lcl}\Delta u+2\varepsilon e^u & = & 0,\\ u(s,0) & = & f(s),\\ u_t(s,0) & = & ce^{f(s)/2}, \qquad c\in\mathbb{R},\end{array}\right. \tag{3.3}$$

is given by

$$u=\log\frac{4|g'(z)|^2}{(1+\varepsilon|g(z)|^2)^2}, \qquad z=s+it, \tag{3.4}$$

where here $g(z)$ is the meromorphic extension to an open set $\Omega\subseteq\mathbb{C}$ containing I of $g(s)=\pi(\alpha(s))$, being $\alpha(s)$ a curve in $\mathcal{Q}(\varepsilon)$ with constant curvature $-c/2$ and arclength parameter $v(s)=\int^s e^{f(r)/2}dr$.

Proof: In [GaMi, Theorem 3] the authors proved that if $\alpha(s)$ is a curve in $\mathcal{Q}(\varepsilon)$ with arclength parameter and geodesic curvature given, respectively, by

$$v(s)=\int^s\sqrt{a}(r)dr, \qquad \kappa(s)=-\frac{d(s)}{2a(s)^{3/2}}, \tag{3.5}$$

then the map

$$\phi(s,t)=\frac{4|g'(z)|^2}{(1+\varepsilon|g(z)|^2)^2}, \qquad z=s+it,$$

is the unique solution to the Cauchy problem

$$\left\{\begin{array}{lcl}\Delta(\log\phi)+2\varepsilon\phi & = & 0,\\ \phi(s,0) & = & a(s),\\ \phi_t(s,0) & = & d(s).\end{array}\right. \tag{3.6}$$

Here $a(s),d(s)$ are real analytic with $a(s)>0$ and $g(z)$ is the meromorphic extension of $g(s)=\pi(\alpha(s))$.

In these conditions, let u denote the solution to (3.3), and write $\phi=e^u$, as well as $\phi(s,0)=a(s)$ and $\phi_t(s,0)=d(s)$. It is then clear by (3.3) that $a(s)=e^{f(s)}$ and $d(s)=ce^{3f(s)/2}$. Thus, the curve $\alpha(s)$ in $\mathcal{Q}(\varepsilon)$ that solves (3.6) with these data is the one whose arclength parameter and geodesic curvature are, respectively,

$$v(s)=\int^s e^{f(r)/2}dr, \qquad \kappa=-c/2.$$

Noting that $u=\log\phi$, we get the desired result.

□

It is interesting to observe here that if we choose two different curves α_1, α_2 in $\mathcal{Q}(\varepsilon)$ with constant curvature $-c/2$ and the same arclength parameter, they differ by an isometry preserving orientations of $\mathcal{Q}(\varepsilon)$. Thus $g_i(s) = \pi(\alpha_i(s))$ differ only by a linear fractional transformation (2.2), which has no effect on the solution u to the problem.

An important observation now is that the curves with constant curvature and prescribed arclength parameter in $\mathcal{Q}(\varepsilon)$ are well known, and thereby we can write an explicit formula in terms of complex analysis for the solution to the Cauchy problem (3.3).

Specifically, if $\varepsilon = 1$, the only (up to isometries preserving orientations) curve in $\mathbb{S}^2 \subset \mathbb{R}^3$ with arclength parameter $v(s)$ and constant geodesic curvature κ is

$$\alpha(s) = \left(\sqrt{1-\varrho^2}, \varrho\cos(v(s)/\varrho), \varrho\sin(v(s)/\varrho)\right), \qquad \varrho := \frac{\mathrm{sg}(\kappa)}{\sqrt{1+\kappa^2}}. \tag{3.7}$$

When $\varepsilon = 0$ and so $\mathcal{Q}(\varepsilon) = \mathbb{R}^2$, the curve $\alpha(s)$ in the above conditions is

$$\alpha(s) = \left(\frac{1}{\kappa}\cos(\kappa\, v(s)), \frac{1}{\kappa}\sin(\kappa\, v(s))\right) \quad \text{or} \quad \alpha(s) = (v(s), 0)$$

depending on whether $\kappa \neq 0$ or $\kappa = 0$.

At last, when $\varepsilon = -1$, the only (up to isometries preserving orientations) curve $\alpha(s)$ in $\mathbb{H}^2$ with arclength parameter $v(s)$ and geodesic curvature κ is

$$\begin{cases} \alpha(s) = \left(\sqrt{1+\varrho^2}, \varrho\cos(v(s)/\varrho), \varrho\sin(v(s)/\varrho)\right), \quad \varrho := \dfrac{\mathrm{sg}(\kappa)}{\sqrt{\kappa^2-1}} & \text{if} \quad |\kappa| > 1, \\ \alpha(s) = (1 + v(s)^2/2, v(s)^2/2, -\mathrm{sg}(\kappa)\, v(s)) & \text{if} \quad |\kappa| = 1, \\ \alpha(s) = \left(\varrho\cosh(v(s)/\varrho), \varrho\sinh(v(s)/\varrho), -\mathrm{sg}(\kappa)\sqrt{\varrho^2-1}\right), \quad \varrho := \dfrac{1}{\sqrt{1-\kappa^2}} & \text{if} \quad |\kappa| < 1. \end{cases}$$

Once at this point, we can use Theorem 2 and the above description in order to classify the solutions to (1.1).

The solution for $\Delta u + 2e^u = 0$

Let $u : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ denote a solution to (1.1) for $a = 2$. The previous discussion shows that u can be recovered as (2.3) in terms of a locally univalent meromorphic map g in $\mathbb{C}$. In addition, by Theorem 2 and (3.7) we get that, on the real axis,

$$g(s) = \alpha \exp\left(i \int^s \mu(r)dr\right), \tag{3.8}$$

where

$$\alpha := \frac{2\,\mathrm{sg(c)}}{|c| - \sqrt{\mathrm{c}^2+4}}, \qquad \mu(s) := -\frac{\mathrm{sg(c)}\sqrt{\mathrm{c}^2+4}}{2}\, e^{f(s)/2}.$$

Let us seek properties of the function $\mu(s)$ that are implied by the properties of $g(z)$. Firstly, observe that $g'(s)/g(s) = i\mu(s)$. Thus, the real analytic function μ can be extended to a meromorphic function in $\mathbb{C}$ satisfying $g'(z)/g(z) = i\mu(z)$. Hence, $\mu(z) \neq 0$ at every point and the poles of $\mu(z)$ are of two different kinds.

(A) $\mu(z)$ has poles at the poles of $g(z)$. Since g is locally univalent then it has only simple poles. Thus, if $z_0 \in \mathbb{C}$ is any of such poles, then μ has at z_0 a simple pole of residue i.

(B) $\mu(z)$ has poles at the zeros of $g(z)$. These poles are simple and of residue $-i$, again by the fact that $g(z)$ can only have simple zeros.

Once here, let us define $h(s) : \mathbb{R} \to \mathbb{R}$ by $h(s) = 1/\mu(s)$. The properties of $\mu(s)$ discussed above yield the following properties for $h(s)$.

1. $h(s)$ admits an entire extension $h(z)$, i.e. $h(s) \in \mathcal{A}$.
2. $h(z)$ only has simple zeros, located at the poles of $\mu(z)$.

Now, if $z_0 \in \mathbb{C}$ is a pole of $\mu(z)$ of type **(A)** (resp. **(B)**), then z_0 is a zero of $h(z)$ with $h'(z_0) = -i$ (resp. $h'(z_0) = i$). This proves that any solution to (1.1) for $a = 2$ is of the form given by (2.3), (2.4), as wished. It is remarkable that u can actually be extended to be a solution of $\Delta u + 2e^u = 0$ globally defined in $\mathbb{R}^2$.

Conversely, let $h(s) \in \mathcal{A}$ with $h(s) \neq 0$ and such that its entire extension $h(z)$ only has simple zeros with $h'(z_0) = \pm i$ at any of them, and let us define $g(z)$ by (2.4). It is clear that g is well defined and holomorphic away from the zeros of $h(z)$. Now let $z_0 \in \mathbb{C}$ denote one of such zeros. Then it is a simple pole of residue $\pm i$ of $\mu(z) := 1/h(z)$. Hence

$$i \int^z \mu(\zeta) d\zeta = \pm \log(z - z_0) + \quad \text{a holomorphic part}$$

around z_0. With this, by (2.4) we see that

$$g(z) = (z - z_0)^{\pm 1} \cdot \big(\text{ a non-zero holomorphic part around } z_0).$$

As a conclusion, $g(z)$ is a meromorphic function globally defined in $\mathbb{C}$, having poles at the zeros of h with $h'(z_0) = -i$. In addition, $g'(z) \neq 0$ because $g'(z)/g(z) = i/h(z) \neq 0$. Hence, Theorem 1 ensures that the map $u : \mathbb{R}^2 \to \mathbb{R}$ given by (2.3) is a global solution to the Liouville equation $\Delta u + 2e^u = 0$. Finally, observe that $g(s)$ takes its values in the circle centered at the origin and of radius $|\alpha| = -2/(|c| - \sqrt{c^2 + 4})$, and by (3.7) it is the stereographic projection of a curve in $\mathbb{S}^2$ of constant curvature $\kappa = -c/2$. Hence, by Theorem 2, the map u fulfills the Neumann boundary condition, and so is a solution to (1.1).

The solution for $\Delta u = 0$

The case $c = 0$ is an immediate consequence of the Cauchy-Riemann equations at points of the real axis.

The situation when $c \neq 0$ is analogous to the case $a = 2$ explained previously. The main difference is that this time the map g given by (2.1) is holomorphic in $\mathbb{C}_+$. So, if we go through the proof of the $a = 2$ case, we easily deduce that this condition is equivalent to the fact that $h'(z_0) = i$ (and not $h'(z_0) = -i$) at each zero $z_0 \in \mathbb{C}_+$ of the entire function $h(z)$. We omit the details.

The solution for $\Delta u - 2e^u = 0$

Let u denote a solution to (1.1) for $a = -2$. Then we know the existence of a locally univalent holomorphic map g from $\mathbb{C}_+$ into the unit disk $\mathbb{D}$ such that, on $\mathbb{R}^2_+ \equiv \mathbb{C}_+$,

$$u = \log \frac{4|g'(z)|^2}{(1-|g(z)|^2)^2}.$$

In addition, we saw previously that g can be extended as a locally univalent meromorphic map to $\mathbb{C}$. And, since u is well defined in $\mathbb{R}$ and $g'(z) \neq 0$ there, we obtain $|g(z)| < 1$ in an open set $\Omega \subset \mathbb{C}$ containing $\overline{\mathbb{C}_+}$. This fact will allow us to use Theorem 2.

Once here, let us prove that the solution to (1.1) is given by the casuistic exposed in Section 2.

In [Zha] it was shown that the problem (1.1) for $a = -2$ has no solution if $c \geq 0$. First of all, we shall show that, actually, the problem does not have a solution if $c > -2$.

The case $|c| < 2$: Take $c \in (-2, 2)$, and consider the existence of a solution u to (1.1) for $a = -2$ and c. Then u satisfies (2.6), where $g(z)$ is the holomorphic extension of $g(s) := \pi(\alpha(s))$ and $\alpha(s)$ is a curve in $\mathbb{H}^2$ of prescribed arclength and of constant curvature $\kappa = -c/2$ (hence, $|\kappa| < 1$).

But it is now easy to check from the description of the curves of constant curvature in $\mathbb{H}^2$ explained previously that $g(\mathbb{R})$ is contained in the circle C_κ of $\mathbb{C}$ centered at $i\sqrt{1-\kappa^2}/\kappa$ and of radius $1/|\kappa|$ if $\kappa \neq 0$, or in the real axis C_0 if $\kappa = 0$.

Now, since $g(\mathbb{R})$ lies in a circle or a line then by the Schwarz reflection principle we have $g(\bar{z}) = J(g(z))$ for all $z \in \mathbb{C}$, where J denotes the inversion with respect to C_κ. Therefore, as $g(\mathbb{C}_+ \cup \mathbb{R}) \subset \mathbb{D}$ then if $w_0 \in C_\kappa$ with $|w_0| \geq 1$ there exists no $z_0 \in \mathbb{C}$ such that $g(z_0) = w_0$. Thus, g is a meromorphic map in $\mathbb{C}$ not containing infinitely many points in its image. Hence, g must be constant, which is a contradiction.

The case $c = -2$: Let u denote a solution to (1.1) for $a = -2$, $c = -2$. Then we know the existence of a locally univalent meromorphic map g in $\mathbb{C}$ such that (2.6) holds in $\mathbb{C}_+ \cup \mathbb{R}$. Moreover, as we saw previously, $g(s) = \pi(\alpha(s))$ where $\alpha(s)$ is a curve in $\mathbb{H}^2$ with geodesic curvature $\kappa = 1$ and prescribed arclength parameter. Specifically, if $f(s) := u(s, 0)$, then

$$\alpha(s) = \left(1 + h(s)^2/2, h(s)^2/2, -h(s)\right), \qquad h(s) := \int_{s_0}^{s} e^{f(r)/2} dr.$$

Consequently, $g(s) = h(s)/(h(s) + 2i)$ and so $h(s)$ admits a holomorphic extension $h(z)$ in $\mathbb{C}$ with

$$g(z) = \frac{h(z)}{h(z) + 2i}. \tag{3.9}$$

On the other hand, $|g| < 1$ in $\mathbb{C}_+$ or equivalently $\mathrm{Im} h > -1$ in $\mathbb{C}_+$. In addition, $\mathrm{Im} h = 0$ on $\mathbb{R}$. Therefore, as we will prove later in Lemma 3, we obtain that $\mathrm{Im} h > 0$ in $\mathbb{C}_+$. That is, $h(\mathbb{C}_+) \subset \mathbb{C}_+$, $h(\mathbb{C}_-) \subset \mathbb{C}_-$ and $h(\mathbb{R}) \subset \mathbb{R}$ with $h'(s) > 0$.

By the little Picard theorem, an entire function is a polynomial, or it assumes every value in $\mathbb{C}$ with at most one exception infinitely many times. In our situation, $h(s)$

is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and $h(\mathbb{C}_\pm) \subset \mathbb{C}_\pm$. So, $h(z)$ must be a one-to-one polynomial, i.e. $h(z) = h_0 z + h_1$ for suitable real numbers h_0, h_1 with $h_0 > 0$. Consequently,

$$u(s,t) = \log\left(\frac{h_0^2}{(1+h_0 t)^2}\right),$$

as we wanted to prove.

In order to complete the $c = -2$ case we prove the following result,

Lemma 3 *Let $D \subset \mathbb{R}^2_+$ be a discrete set and $u \in C^0(\overline{\mathbb{R}^2_+}\backslash D) \cap C^2(\mathbb{R}^2_+\backslash D)$ a non constant function. Let us assume $\Delta u = 0$ on $\mathbb{R}^2_+\backslash D$, $u = 0$ on $\partial\mathbb{R}^2_+$ and u is bounded from below in $\overline{\mathbb{R}^2_+}\backslash D$. Then $u(s,t) > 0$ for all $(s,t) \in \mathbb{R}^2_+\backslash D$.*

Proof: Let $z_0 \in \mathbb{C}_+\backslash D \equiv \mathbb{R}^2_+\backslash D$ and F a Möbius transformation such that $F(\mathbb{R})$ is the unit circle except the number 1 and $F(z_0) = 0$ (so, $F(\mathbb{C}_+) = \mathbb{D}$).

Then, $u_0 = u \circ F^{-1} \in C^0\left(\overline{\mathbb{D}}\backslash(F(D) \cup \{1\})\right)$ is a harmonic map in $\mathbb{D}\backslash F(D)$ such that $u_0 = 0$ on $\partial\mathbb{D}\backslash\{1\}$. In addition, by hypothesis there exists $r_0 < 0$ such that $u_0(z) \geq r_0$ for any $z \in \mathbb{D}\backslash F(D)$.

Let $\varphi_n : \partial\mathbb{D} \longrightarrow [r_0, 0]$ be a sequence of continuous functions such that

i) $\varphi_n(e^{i\theta}) = 0$ if $|\theta| \geq 1/n$, with $\theta \in [-\pi, \pi[$,

ii) $\varphi_n(1) = r_0$.

Consider $u_n \in C^0(\overline{\mathbb{D}}) \cap C^2(\mathbb{D})$ the solution to the Dirichlet problem $\Delta u_n = 0$ in $\mathbb{D}$ and $u_n = \varphi_n$ on $\partial\mathbb{D}$.

Now, we observe that given $w_0 \in F(D)$ then u_0 tends to $+\infty$ when z tends to w_0 or u_0 has a removable singularity at w_0 because u_0 is bounded from below. Then, from the maximum principle for harmonic maps

$$u_n(z) \leq u_0(z), \qquad \forall z \in \overline{\mathbb{D}}\backslash(F(D) \cup \{1\}).$$

On the other hand, from the mean value property and i), ii)

$$u_n(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(e^{i\theta})\, d\theta \to 0 \qquad \text{when } n \to +\infty.$$

Therefore, $u_0(0) \geq 0$ or equivalently $u(z_0) \geq 0$. That is, u is a non negative function in $\mathbb{C}_+\backslash D$. Moreover, a harmonic function cannot have local minima at interior points unless u is constant. Hence, $u(z)$ must be positive for all $z \in \mathbb{C}_+\backslash D$.

□

The case $c < -2$**:** Let u be a solution to (1.1) for $a = -2$ and $c < -2$. Then u can be recovered as (2.6) in terms of a meromorphic map g on $\mathbb{C}$ such that $g(\overline{\mathbb{C}_+}) \subset \mathbb{D}$. Moreover, $g(s) = \pi(\alpha(s))$ where $\alpha(s)$ is a curve in $\mathbb{H}^2$ with geodesic curvature $\kappa = -c/2 > 1$.

Then reasoning as in the $a = 2$ case we have that $g(z)$ is given as (2.7), where $h(s) \in \mathcal{A}$ satisfies $h(s) \neq 0$ at every point and its entire extension $h(z)$ only has simple zeros with $h'(z_0) = i$ at each of the zeros lying in $\mathbb{C}_+$. In addition, $|g| < 1$ in $\overline{\mathbb{C}_+}$, so (2.5) is satisfied.

Conversely, if we follow the proof for the $a = 2$ case, we only need to show that $|g(z)| < 1$ in $\overline{\mathbb{C}_+}$ for $g(z)$ given as (2.7). Since $|g(s)| = 2/(-c + \sqrt{c^2 - 4})$ for any $s \in \mathbb{R}$ and we assume (2.5), then the harmonic function

$$\widetilde{u}(z) = \log\left(\frac{2}{(-c + \sqrt{c^2 - 4})|g(z)|}\right)$$

is in the conditions of Lemma 3. Here, $\widetilde{u}$ is defined in $\overline{\mathbb{C}_+}$ except at the points where g vanishes. Thus, $\widetilde{u}(z) \geq 0$ by Lemma 3, or equivalently, $|g(z)| \leq 2/(-c + \sqrt{c^2 - 4}) < 1$ in $\overline{\mathbb{C}_+}$ as we wanted to prove.

Remark 4 *Following the proofs of the cases $c = -2$ and $c < -2$ for $a = -2$ it is easy to show that there exists no solution for the cases $c = 2$ and $c > 2$, respectively. This fact can be used to give an alternative proof to Zhang's non existence result* [Zha].

The method that we introduce here also gives an alternative proof of the original solution to the problem (1.1)-(1.2) when $a = 2$, given in [Zha] and [LiZh], as we explain next (as a matter of fact, it also works with slight modifications for the cases $a = 0, -2$):

Corollary 5 ([Zha], Theorem 3.1) *Any solution to the problem* (1.1)-(1.2) *for the case $a = 2$ takes the form*

$$u(s,t) = \log\left(\frac{4\lambda^2}{(\lambda^2 + (s - s_0)^2 + (t - t_0)^2)^2}\right), \tag{3.10}$$

for some $\lambda > 0$, $s_0 \in \mathbb{R}$ and $t_0 = (c/2)\lambda$.

Proof: Our solution to the problem (1.1) for $a = 2$ shows that any solution u to this problem can be written as (2.3) for some locally univalent meromorphic function g globally defined on $\mathbb{C}$. Besides, we have proved that, up to a conformal transformation that has no effect on the value of the solution u, the image via g of $\mathbb{R} \cup \{\infty\}$ is the circle centered at the origin and of radius $|r|$, where

$$r = \frac{2\,\mathrm{sg(c)}}{|c| - \sqrt{c^2 + 4}}.$$

Thus, by Schwarz's reflection principle,

$$g(z) = \frac{r^2}{\overline{g(\bar{z})}}.$$

This implies after a calculation that for every $z \in \mathbb{C}_- \equiv \mathbb{R}^2_-$ it holds

$$\frac{4|g'(z)|^2}{(1+|g(z)|^2)^2} = \frac{4\frac{1}{r^2}|g'(\bar{z})|^2}{\left(1+\frac{1}{r^2}|g(\bar{z})|^2\right)^2}.$$

Thus, from (2.3) we get

$$\int_{\mathbb{R}^2_-} e^u = \frac{1}{r^2}\int_{\mathbb{R}^2_+} e^u.$$

Consequently, (1.2) is equivalent to $\int_{\mathbb{R}^2} e^u < \infty$, which simply means that the pullback metric of the sphere via g on $\mathbb{C}$ has finite area. Therefore, (1.2) is equivalent to the fact that g does not have an essential singularity at ∞.

Summarizing, g must be a locally univalent rational function (a Möbius transformation) that moreover maps $\mathbb{R} \cup \{\infty\}$ into $\mathbb{S}^1(|r|)$. Besides, by (3.7), the circle is covered counterclockwise when $c \geq 0$ and clockwise when $c < 0$. Hence,

$$g(z) = re^{i\theta}\frac{\zeta - a}{1-\bar{a}\zeta}, \qquad \zeta := \frac{\text{sg}(c)iz+1}{-\text{sg}(c)iz+1},$$

where $\theta \in \mathbb{R}$ and $a \in \mathbb{C}$ with $|a| < 1$. Once here, writing $a = a_1 + ia_2$ and $z = s + it$, a direct but long computation gives

$$\frac{|g'(z)|^2}{(1+|g(z)|^2)^2} = \left(\frac{4r(1-|a|^2)}{(1+r^2)((1+a_1)^2+a_2^2)((s-s_0)^2+(t-t_0)^2+A)}\right)^2,$$

where

$$s_0 := \frac{2a_2\text{sg}(c)}{(1+a_1)^2+a_2^2}, \qquad t_0 := \frac{(1-|a|^2)(r^2-1)\text{sg}(c)}{((1+a_1)^2+a_2^2)(1+r^2)}$$

and

$$A := \frac{4(1-|a|^2)^2r^2}{((1+a_1)^2+a_2^2)^2(1+r^2)^2}.$$

That is, u has the form (3.10) for

$$\lambda = \frac{-2\text{sg}(c)(1-|a|^2)r}{((1+a_1)^2+a_2^2)(1+r^2)} > 0 \quad \text{and} \quad \frac{t_0}{\lambda} = \frac{1-r^2}{2r} = \frac{c}{2}.$$

□

4 The punctured disc problem

Let us start by solving the Neumann problem (1.4) in the punctured disc $\mathbb{D}^*$ by means of the solution to (1.1) exposed in Section 3. For that, consider the map $\Phi : \mathbb{R}^2_+ \to \mathbb{D}^*$ given by $\Phi(s,t) = (e^{-t}\cos s, e^{-t}\sin s)$. Thus, Φ is a conformal map that is 2π-periodic in the s variable. Moreover, Φ defines a conformal isomorphism between the quotient cylinder $\mathbb{R}^2_+/(2\pi\mathbb{Z})$ and $\mathbb{D}^*$. With this, we have the following elementary lemma (see [BHL] for related results).

Lemma 6 *Let* $u(x,y) : \mathbb{D}^* \cup \mathbb{S}^1 \to \mathbb{R}$ *be a solution to* (1.4)*. Then the map*

$$w(s,t) := -2t + u(\Phi(s,t)) \tag{4.1}$$

is a solution to (1.1) *such that* $w(s,0)$ *is* 2π*-periodic.*

Conversely, let $u(s,t) : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ *denote a solution to* (1.1) *such that* $u(s,0)$ *is* 2π*-periodic. Then* u *is single-valued in* $\mathbb{R}^2_+/(2\pi\mathbb{Z})$ *and the map* $w(x,y) : \mathbb{D}^* \cup \mathbb{S}^1 \to \mathbb{R}$ *given by*

$$w(x,y) = u\left(\Phi^{-1}(x,y)\right) - \log(x^2+y^2) \tag{4.2}$$

is a solution to (1.4)*.*

Proof: The first part is a straightforward computation, using that $-\partial u/\partial \nu = xu_x + yu_y$ on $\partial\mathbb{D} = \mathbb{S}^1$ for the exterior unit normal ν.

For the converse part, we observe that if $u(s,t)$ is a solution to the Cauchy problem (3.3), then u is 2π-periodic in the s variable if and only if $f(s) = u(s,0)$ is 2π-periodic. Thus, as $u(s,0)$ is 2π-periodic in the present situation, the solution $u(s,t)$ to (1.1) is well defined in $\mathbb{R}^2_+/(2\pi\mathbb{Z})$. At last, it is immediate that (4.2) is the inverse expression of (4.1), and so we are done.

$\square$

The following result is a direct consequence of Lemma 6 and the solution to (1.1).

Corollary 7 *Let* $u(x,y)$ *denote a solution to* (1.4) *for* $a=-2$ *and* $c \geq -2$*. Then* $c=-2$ *and* u *is the radial solution*

$$u(r) = 2\log\left(\frac{h_0}{r(1-h_0\log r)}\right), \qquad 0<r<1, \quad h_0>0,$$

where here $r := \sqrt{x^2+y^2}$*.*

For the remaining cases, Lemma 6 indicates that for solving (1.4) we simply need to analyze what the 2π-periodicity condition on $f(s) = u(s,0)$ means in the solution to (1.1) exposed in Section 3. In order to do so, define $\mathcal{F}$ as the class of 2π-periodic elements of the class $\mathcal{A}$ defined in Section 2. Equivalently, $\mathcal{F}$ consists of the restrictions to $\mathbb{R}$ of the entire 2π-periodic functions in $\mathbb{C}$ that take real values on the real axis.

As we exposed in Section 2, any solution to (1.1) when $a=0,2$ or when $a=-2$ and $c<-2$ is constructed in terms of a certain real analytic function $h(s) : \mathbb{R} \to \mathbb{R}$ so that $h(s) \in \mathcal{A}$. Moreover, it is easy to verify in each case that $h(s)$ is 2π-periodic if and only if $f(s) := u(s,0)$ is 2π-periodic. This yields the following conclusion.

Corollary 8 *Let* u *denote a solution to* (1.1) *for* $a=0,2$ *or for* $a=-2$ *and* $c<-2$*. Consider in addition the function* $h(s) \in \mathcal{A}$ *in terms of which* u *can be described, as exposed in Section 2. Then* u *defines via Lemma 6 a solution to* (1.4) *if and only if* $h(s) \in \mathcal{F}$*.*

From now on, we will study the problem (1.4) under the additional condition

$$\int_{\mathbb{D}^*} e^u < +\infty \tag{4.3}$$

Let us point out that, although we have seen that (1.4) and (1.1) are intimately related, it is not true that the solutions to (1.4) that verify (4.3) come from solutions to (1.1) satisfying (1.2) via Corollary 8.

To start with, we have

Proposition 9 *The problem* (1.4) *with the integral condition* (4.3) *does not have a solution for $a = c = 0$.*

Proof: Let $u(x, y)$ denote a solution to (1.4) with $a = c = 0$. By Corollary 8 and the solution to (1.1) for $a = c = 0$, there exists an entire 2π-periodic function $h(z)$ with $h(s) \in \mathbb{R}$, such that

$$u(x, y) = \text{Re}(h(-i \log \xi)) - \log |\xi|^2, \qquad \xi := x + iy.$$

Here, observe that $F(\xi) := h(-i \log \xi)$ is well defined and holomorphic in $\mathbb{C}^*$, since h is 2π-periodic. In addition, it is clear that

$$\int_{\mathbb{D}^*} e^u = \int_{\mathbb{D}^*} \left| \frac{e^{F(\xi)}}{\xi^2} \right|.$$

Thus, by (4.3) we see that e^F must extend holomorphically across the origin with value 0 there. But this is not possible, since F is single valued in $\mathbb{C}^*$.

□

At last, we are in the position to describe all the solutions to (1.4) that satisfy (4.3). For that, we will rule out without loss of generality the case where $u \in C^2(\mathbb{D})$ (see Remark 11 below).

Theorem 10 *Let $u(x, y)$ denote a solution to* (1.4) *satisfying the integral condition* (4.3)*, and assume that u does not extend smoothly across the origin. Then u is radially symmetric, i.e. $u = u(r)$ where $r := \sqrt{x^2 + y^2}$, and it is given by:*

1. *If $a = 2, c \leq 0$, or if $a = 0, c < 0$, or if $a = -2, c < -2$, then*

$$u(r) = 2 \log \frac{2R\beta\, r^{\beta-1}}{1 + \frac{a}{2} R^2\, r^{2\beta}}. \tag{4.4}$$

2. *If $a = 2$ and $c > 0$, then*

$$u(r) = 2 \log \frac{2R\beta\, r^{1-\beta}}{1 + R^2\, r^{-2\beta}}. \tag{4.5}$$

3. If $a=-2=c$, then

$$u(r)=2\log\left(\frac{h_0}{r(1-h_0\log r)}\right),\qquad h_0>0. \tag{4.6}$$

Here β is positive with $\beta\neq 1$, and $R>0$ is given by

$$\left\{\begin{array}{lll} R=\dfrac{2}{\sqrt{c^2+4}-|c|} & \text{if} & a=2,\\ R=\dfrac{2}{|c|} & \text{if} & a=0,\\ R=\dfrac{2}{\sqrt{c^2-4}+|c|} & \text{if} & a=-2. \end{array}\right. \tag{4.7}$$

The problem does not have a solution in the remaining cases, i.e. if $a=0, c\geq 0$ or if $a=-2, c>-2$.

Proof: The cases $a=-2$, $c\geq -2$ and $a=c=0$ are already covered by Corollary 8 and Proposition 9, taking into account that (4.6) trivially satisfies (4.3). For the remaining cases, assume that $u(x,y)\in C^2(\mathbb{D}^*)$ is a solution to (1.4), (4.3), and let

$$w(s,t):=-2u+u(e^{-t}\cos s, e^{-t}\sin s)$$

denote the 2π-periodic solution to (1.1) indicated in Lemma 6. It follows then from the solution to (1.1) exposed in Section 2 for $a=0,2$ and for $a=-2, c<-2$ that

$$w(s,t)=\log\frac{4|g'(z)|^2}{(1+\frac{a}{2}|g(z)|^2)^2},\qquad z=s+it.$$

Here g is meromorphic with $|g(s)|=R$ for $s\in\mathbb{R}$, where $R>0$ is given in terms of a,c by (4.7).

Define next $G(\zeta):=g(-i\log\zeta)$, which is a locally one-to-one multivalued meromorphic function on $\mathbb{C}^*$. Besides, again by Lemma 6, it is clear that

$$u(\zeta)=\log\frac{4|G'(\zeta)|^2}{(1+\frac{a}{2}|G(\zeta)|^2)^2},\qquad \zeta=x+iy. \tag{4.8}$$

From there, we can consider the holomorphic function Q^* on $\mathbb{D}^*$,

$$Q^*=u_{\zeta\zeta}-\frac{1}{2}u_\zeta^2=\left(\frac{G''}{G'}\right)'-\frac{1}{2}\left(\frac{G''}{G'}\right)^2=\{G,\zeta\}. \tag{4.9}$$

A straightforward computation analogous to (3.2) (but this time a bit messier) proves that the boundary condition in (1.4) simply means that

$$\text{Im}(\zeta^2 Q^*(\zeta))=0,\qquad \text{on } \mathbb{S}^1=\{|\zeta|=1\}.$$

On the other hand, it is a standard argument (see [ChWa] for instance) to show that

$$G(\zeta) = \zeta^{\alpha} F(\zeta), \qquad \alpha \in [0,1), \tag{4.10}$$

for some constant α, where F is a single valued meromorphic function in $\mathbb{C}^*$. Moreover, by [ChWa, Lemma 4] we see that the integral finiteness condition (4.3) implies that F has at worst a pole at the origin, i.e. it cannot have an essential singularity at 0. We remark that this condition is immediate when $a = -2$, since in that case we know that $|G| < 1$.

Thereby, we can conclude that

$$\frac{G''(\zeta)}{G'(\zeta)} = \frac{\alpha(\alpha-1)F(\zeta) + 2\alpha\zeta F'(\zeta) + \zeta^2 F''(\zeta)}{\alpha\zeta F(\zeta) + \zeta^2 F'(\zeta)}$$

has at worst a pole of order 1 at the origin. That is, Q^* has at worst a pole of order two at 0. Consequently, $\zeta^2 Q^*(\zeta)$ is holomorphic in $\overline{\mathbb{D}}$, and in addition, we know that $\mathrm{Im}(\zeta^2 Q^*(\zeta)) = 0$ over $\partial\mathbb{D}$. Therefore $\mathrm{Im}(\zeta^2 Q^*(\zeta)) = 0$ on $\mathbb{D}$ and

$$Q^*(\zeta) = \frac{r_0}{\zeta^2}, \qquad \text{for some } r_0 \in \mathbb{R}.$$

Now, by (4.9), the map $G(\zeta)$ is a local solution to $\{G, \zeta\} = r_0\zeta^{-2}$. But this equation is classically known to have a unique solution (up to Möbius transformations), which is

1. $G(\zeta) = \log\zeta$ if $r_0 = 1/2$.
2. $G(\zeta) = \zeta^{\beta}$, where $\beta^2 = 1 - 2r_0$, in case $r_0 \neq 1/2$.

Once here, we must stress that, by (4.10), the modulus $|G(\zeta)|$ is single valued on $\mathbb{D}^*$. This excludes the first case, as well as the situation in which β is not a real number, i.e. the case in which $r_0 \geq 1/2$. Thus, G is of the form

$$G(\zeta) = \mathcal{M}(\zeta^{\beta}), \tag{4.11}$$

where $\mathcal{M}$ is a Möbius transformation, and $\beta > 0$. Moreover, we know that G is locally injective, and that $|G(\zeta)| = R$ when $|\zeta| = 1$.

Assume first that $\beta \neq 1$. Then ζ^{β} is locally one-to-one except around 0 and ∞. Thus, as this property is also true for G, by (4.11) we have $\mathcal{M}(0) = 0$ or $\mathcal{M}(0) = \infty$ (this second possibility does not occur if $a = -2$, since $|G| < 1$ in that case). And as $|G(\zeta)| = R$ when $|\zeta| = 1$, we also have $|\mathcal{M}(\zeta)| = R$ when $|\zeta| = 1$. Thereby,

$$\mathcal{M}(\zeta) = Re^{i\theta_0}\zeta \qquad \text{or} \qquad \mathcal{M}(\zeta) = Re^{i\theta_0}\zeta^{-1}$$

for some $\theta_0 \in \mathbb{R}$, where the second possibility does not happen if $a = -2$. At last, by (4.8) and (4.11), the solution $u \in C^2(\mathbb{D}^*)$ is given by (4.4), by (4.5), or by

$$u(r) = 2\log(2\beta R r^{-(\beta+1)}), \tag{4.12}$$

which corresponds to the case $a = 0$ and $\mathcal{M}(\zeta) = Re^{i\theta_0}\zeta^{-1}$. It is immediate to check that (4.4) and (4.5) verify (4.3), while (4.12) does not. So, (4.12) must be suppressed from the final classification theorem.

Finally, we need to rule out the case $\beta = 1$. Indeed, in that situation we have that G is itself a Möbius transformation with $|G(\zeta)| = R$ whenever $|\zeta| = 1$. Thus, the solution $u \in C^2(\mathbb{D}^*)$ given explicitly by (4.8) extends smoothly across the origin, a condition that was supposed not to hold in the beginning. This completes the proof.

$\square$

Remark 11 *If in* (1.4) *we assume that* $u \in C^2(\mathbb{D})$, *then* (4.3) *automatically holds, and the problem was solved in [HaWa]. Let us nonetheless remark that the proof of Theorem 10 actually provides an alternative proof of the classification by Hang and Wang of the solutions to* (1.4) *in* $\mathbb{D}$. *Indeed, the smoothness of* u *at the origin simply corresponds to the case* $\beta = 1$ *treated in the last part of the proof.*

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